Determine The Dimensions Of A Rectangular Solid (with A Square Base) With Maximum Volume If Its Surface

Determine The Dimensions Of A Rectangular Solid (with A Square Base) With Maximum Volume If Its Surface Area Is Fixed

In optimization problems involving three-dimensional objects, a common challenge is to find the dimensions that maximize or minimize a certain property—such as volume—while adhering to given constraints. One classic problem involves determining the dimensions of a rectangular solid with a square base that yields the maximum volume when the surface area is fixed. This problem combines principles from calculus, geometry, and algebra, and it offers valuable insights into how to approach constrained optimization problems. In this article, we will explore the problem systematically, develop the mathematical formulation, and derive the conditions for the maximum volume.

Understanding the Problem Setup

Defining the Variables

- Let the side length of the square base be x.
- Let the height of the rectangular solid be h.

Given Constraints

- The surface area of the rectangular solid is fixed at a value S.
- The goal is to determine the values of x and h that maximize the volume V.

Mathematical Formulation

- Volume of the solid: $V = x^2h$
- Surface area of the solid: $S = 2x^2 + 4xh$ (since the base and top are squares and the sides are rectangles)

The problem reduces to:

> Maximize $V = x^2h$ subject to the constraint $2x^2 + 4xh = S$.

Developing the Mathematical Model

Expressing One Variable in Terms of the Other

- From the surface area constraint, solve for h:

$$h = (S - 2x^2) / (4x)$$

- Note: x must be positive and such that $S - 2x^2 > 0$ (to keep h positive).

Expressing Volume as a Function of One Variable

- Substitute h into the volume formula:

$$V(x) = x^2 \quad [(S - 2x^2) / (4x)] = (x^2 \quad (S - 2x^2)) / (4x) = (x(S - 2x^2)) / 4$$

- Simplify:

$$V(x) = (x \quad S - 2x^3) / 4$$

- To maximize volume, focus on the function:

$$V(x) = (xS - 2x^3) / 4$$

Finding the Critical Points for Maximum Volume

Differentiate the Volume Function

- Compute the derivative of $V(x)$ with respect to x:

$$V'(x) = (S - 6x^2) / 4$$

Set the Derivative to Zero

- Critical points occur when $V'(x) = 0$:

$$(S - 6x^2) / 4 = 0 \Rightarrow S - 6x^2 = 0 \Rightarrow 6x^2 = S \Rightarrow x^2 = S/6$$

- Take the positive root (since length must be positive):

$$x = \mathrm{sqrt}(S/6)$$

Determine Corresponding Height

- Substitute *x* back into the expression for *h*:

$$h = (S - 2x^2) / (4x) = (S - 2(S/6)) / (4\ \sqrt{S/6})$$

- Simplify numerator:

$$S - 2(S/6) = S - (S/3) = (3S/3) - (S/3) = (2S/3)$$

- So,

$$h = (2S/3) / (4\ \sqrt{S/6})$$

- Simplify denominator:

$$4\ \sqrt{S/6} = 4\ \sqrt{S} / \sqrt{6}$$

- Therefore,

$$h = (2S/3)\ (\sqrt{6} / (4\ \sqrt{S})) = (2S/3)\ (\sqrt{6} / (4\ \sqrt{S}))$$

- Simplify numerator:

$$(2S/3)\ \sqrt{6} = (2/3)\ S\ \sqrt{6}$$

- Simplify the entire expression:

$$h = [(2/3)\ S\ \sqrt{6}] / [4\ \sqrt{S}] = [(2/3)\ \sqrt{6}\ S] / [4\ \sqrt{S}]$$

- Cancel $\sqrt{S}$:

$$h = [(2/3)\ \sqrt{6}\ S] / [4\ \sqrt{S}] = [(2/3)\ \sqrt{6}\ \sqrt{S}] / 4$$

- Final simplified form:

$$h = (\ (2/3)\ \sqrt{6}\ \sqrt{S}\ ) / 4 = (\ \sqrt{6}\ \sqrt{S}\ ) / (6)$$

Summary of Optimal Dimensions

- Optimal side length of the square base:

$$x_{max} = \sqrt{\frac{S}{6}}$$

- Corresponding height:

$$h_{max} = \frac{\sqrt{6} \cdot \sqrt{S}}{6} = \frac{\sqrt{S}}{\sqrt{6}}$$

Verifying the Nature of the Critical Point

Second Derivative Test

- The second derivative of $V(x)$:

$$V''(x) = -12x / 4 = -3x$$

- Since $x > 0$, $V''(x) < 0$, indicating the critical point corresponds to a maximum.

Interpreting the Results

Implications of the Optimal Dimensions

- The side length of the square base should be proportional to the square root of the fixed surface area.
- The height should be proportional to the square root of the surface area, scaled by the factor $1/\sqrt{6}$.

Practical Considerations

- These results assume the surface area is fixed and positive.
- The dimensions should satisfy physical constraints, such as the height being positive and the base side length being feasible.

Conclusion

The problem of maximizing the volume of a rectangular solid with a square base under a fixed surface area constraint yields elegant solutions rooted in calculus. By expressing one dimension in terms of the other and differentiating, we find that the maximum volume occurs when the side of the base and the height are proportional to the square root of the surface area. Specifically, the optimal dimensions are:

- **Base side length:** $x = \sqrt{\frac{S}{6}}$
- **Height:** $h = \frac{\sqrt{S}}{\sqrt{6}}$

This approach exemplifies the power of calculus in solving constrained optimization problems and provides a clear methodology for determining optimal dimensions in physical and engineering applications.

Frequently Asked Questions

How do I set up the problem to find the dimensions of a rectangular solid with a square base that maximizes volume given a fixed surface area?

To set up the problem, define variables: let x be the side length of the square base and h be the height. Write expressions for volume $V = x^2h$ and surface area $S = 2x^2 + 4xh$. Use the fixed surface area constraint to express h in terms of x, then maximize V with respect to x.

What is the role of calculus in determining the maximum volume of a rectangular solid with a square base under a surface area constraint?

Calculus is used to differentiate the volume function with respect to x after expressing h in terms of x from the surface area constraint. Setting the derivative to zero allows finding the critical points, which correspond to the maximum volume.

How can I verify that the dimensions I find truly maximize the volume for the rectangular solid with a square base?

After finding critical points, use the second derivative test or analyze the behavior of the volume function around these points to confirm they correspond to a maximum. Typically, the second derivative of the volume function will be negative at the maximum point.

Can the method for determining maximum volume be applied to other shapes or constraints?

Yes, the approach of setting up a volume function, applying constraints, and using calculus to find extrema is applicable to various shapes and constraints, such as cylindrical or triangular prisms, by appropriately defining variables and relationships.

What is the practical significance of knowing the dimensions of a rectangular solid with maximum volume under a surface area constraint?

Understanding these dimensions helps optimize material usage, design efficient storage containers, or maximize capacity while minimizing surface material costs in manufacturing and engineering applications.

Additional Resources

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Introduction

Optimizing the dimensions of a three-dimensional object to maximize volume while adhering to certain constraints is a classical problem in calculus and geometric optimization. Specifically, when dealing with a rectangular solid with a square base under a fixed surface area, the goal is to determine the dimensions—length, width, and height—that yield the maximum possible volume.

This problem has practical applications in manufacturing, packaging, and architectural design, where materials are limited or costs are constrained. Understanding how to approach such an optimization provides valuable insights into efficient design and resource management.

Understanding the Problem

The core challenge involves balancing the volume of the rectangular solid against its surface area constraints. The specific parameters are:

- Base shape: Square (meaning length and width are equal)
- Variables:
- Let x = length of the side of the square base
- Let h = height of the solid

- Given:
- Total surface area S (fixed)

- Objective:
- Maximize the volume V

Step 1: Expressing the Geometric Quantities

1.1 Volume of the Rectangular Solid

Since the base is a square:

$$
V = \text{area of base} \times \text{height} = x^2 h
$$

1.2 Surface Area of the Solid

The surface area comprises:

- Base: x^2
- Top: x^2
- 4 Vertical sides: each with area $x \times h$, total $4xh$

Total surface area:

$$
S = 2x^2 + 4xh
$$

Given that S is fixed, this relation links x and h.

Step 2: Formulating the Optimization Problem

The goal is:

$$
\text{Maximize } V = x^2 h
$$

Subject to the constraint:

$$
2x^2 + 4xh = S
$$

Rearranged:

$$
4xh = S - 2x^2
$$

Express h in terms of x:

$$
h = \frac{S - 2x^2}{4x}
$$

Note: Since h must be positive, $S - 2x^2 > 0$.

Step 3: Express the Volume as a Function of a Single Variable

Substitute h into the volume:

$$
V(x) = x^2 \times \frac{S - 2x^2}{4x} = \frac{x (S - 2x^2)}{4}
$$

Simplify:

$$
V(x) = \frac{Sx - 2x^3}{4}
$$

Now, the problem reduces to maximizing:

$$
V(x) = \frac{1}{4} (Sx - 2x^3)
$$

for $x > 0$ such that $S - 2x^2 > 0$.

Step 4: Find Critical Points Using Calculus

Differentiate $V(x)$ with respect to x:

$$
V'(x) = \frac{1}{4} (S - 6x^2)
$$

Set the derivative equal to zero to find critical points:

$$
V'(x) = 0 \Rightarrow S - 6x^2 = 0 \Rightarrow 6x^2 = S
$$

Solve for x:

$$
x^2 = \frac{S}{6} \Rightarrow x = \sqrt{\frac{S}{6}}
$$

Since $x > 0$, we take the positive root.

Step 5: Find the Corresponding Height h

Using the earlier relation:

$$
h = \frac{S - 2x^2}{4x}
$$

Substitute $x^2 = \frac{S}{6}$:

$$
h = \frac{S - 2 \times \frac{S}{6}}{4 \times \sqrt{\frac{S}{6}}}
$$

Simplify numerator:

$$
S - \frac{2S}{6} = S - \frac{S}{3} = \frac{2S}{3}
$$

Now, the denominator:

$$
4 \times \sqrt{\frac{S}{6}} = 4 \times \frac{\sqrt{S}}{\sqrt{6}} = \frac{4 \sqrt{S}}{\sqrt{6}}
$$

Putting it all together:

$$
h = \frac{\frac{2S}{3}}{\frac{4 \sqrt{S}}{\sqrt{6}}} = \frac{2S/3}{4 \sqrt{S}/\sqrt{6}} = \frac{2S/3 \times \sqrt{6}}{4 \sqrt{S}}
$$

Simplify numerator and denominator:

$$
h = \frac{2S \sqrt{6}}{3 \times 4 \sqrt{S}} = \frac{2S \sqrt{6}}{12 \sqrt{S}} = \frac{S \sqrt{6}}{6 \sqrt{S}}
$$

Express S as $\sqrt{S} \times \sqrt{S}$:

$$
h = \frac{\sqrt{S} \times \sqrt{S} \times \sqrt{6}}{6 \sqrt{S}} = \frac{\sqrt{S} \times \sqrt{6}}{6}
$$

Thus:

$$
h = \frac{\sqrt{6} \times \sqrt{S}}{6} = \frac{\sqrt{6S}}{6}
$$

Summary of Optimal Dimensions

- Side length of base:

$$
x^\ast = \sqrt{\frac{S}{6}}
$$

- Height:

$$
h^\ast = \frac{\sqrt{6S}}{6}
$$

Interpretation and Practical Insights

The derived formulas reveal a proportional relationship between the dimensions and the fixed surface area:

- The square base side grows with the square root of the surface area S. Larger surface areas allow for larger base dimensions.
- The height is proportional to the square root of S, scaled by a factor involving $\sqrt{6}$.

This indicates that as the available surface area increases, all dimensions scale up proportionally, maintaining the optimal ratio for maximum volume.

Volume at the Optimal Dimensions

Calculate the maximum volume:

$$
V_{\max} = x^2 h
$$

Substitute:

$$
x^2 = \frac{S}{6}
$$
$$
h = \frac{\sqrt{6S}}{6}
$$

Therefore:

$$
V_{\max} = \frac{S}{6} \times \frac{\sqrt{6S}}{6} = \frac{S \times \sqrt{6S}}{36}
$$

Expressing $\sqrt{6S}$ as:

$$
\sqrt{6S} = \sqrt{6} \times \sqrt{S}
$$

We get:

$$
V_{\max} = \frac{S \times \sqrt{6} \times \sqrt{S}}{36} = \frac{\sqrt{6} \times S \times \sqrt{S}}{36}
$$

Alternatively:

$$
V_{\max} = \frac{\sqrt{6} \times S^{3/2}}{36}
$$

This formula provides the maximum volume achievable given a fixed surface area S.

Additional Considerations and Constraints

- The derivation assumes positive dimensions; thus, the critical point must satisfy:

$$
x > 0, \quad h > 0
$$

- For $h > 0$:

$$
\frac{S - 2x^2}{4x} > 0 \Rightarrow S - 2x^2 > 0 \Rightarrow x^2 < \frac{S}{2}
$$

Since the critical point $x = \sqrt{\frac{S}{6}}$ satisfies this inequality ($\frac{S}{6} < \frac{S}{2}$), the critical point is valid.

- Boundary cases where $x \to 0^+$ or $x \to \sqrt{\frac{S}{2}}^-$ result in negligible or zero volume, confirming that the critical point yields the maximum volume.

Practical Applications and Implications

Understanding the optimal dimensions for a rectangular solid with a square base under a fixed surface area has several real-world applications:

- Packaging Design: Maximize the volume of a box with limited surface material to optimize storage capacity.
- Manufacturing: Design containers that use materials efficiently while maximizing internal space.
- Architectural Elements: Determine proportions of structural elements when material constraints are present.

Designers and engineers can use these formulas to quickly estimate the optimal shape, saving material costs and enhancing efficiency.

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