Determine Whether Each Set Equipped With The Given Operations Is A Vector Space. For Those That Are Not

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Understanding whether a given set, combined with specific operations, constitutes a vector space is a fundamental question in linear algebra. Vector spaces are essential structures that underpin many areas of mathematics, physics, computer science, and engineering. They provide a framework for analyzing linear relationships, transformations, and systems. In this article, we will explore the criteria that define a vector space, analyze various examples, and determine whether the given sets with their operations qualify as vector spaces. For those that do not, we will identify the reasons and the specific axioms they violate.

Fundamentals of Vector Spaces

Before diving into examples, it’s crucial to understand what makes a set a vector space.

Definition of a Vector Space

A vector space V over a field $\mathbb{F}$ (commonly the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$) is a set equipped with two operations:

- Vector addition: $+ : V \times V \to V$
- Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$

These operations must satisfy the following axioms:

1. Closure under addition: For all $u, v \in V$, $u + v \in V$.
2. Commutativity of addition: For all $u, v \in V$, $u + v = v + u$.
3. Associativity of addition: For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
4. Existence of additive identity: There exists an element $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.
5. Existence of additive inverses: For each $v \in V$, there exists $-v \in V$ such that $v + (-v) = \mathbf{0}$.
6. Closure under scalar multiplication: For all $a \in \mathbb{F}$, $v \in V$, $a \cdot v \in V$.
7. Compatibility of scalar multiplication with field multiplication: For all $a, b \in \mathbb{F}$, $v \in V$, $(ab) \cdot v = a \cdot (b \cdot v)$.
8. Identity element of scalar multiplication: For all $v \in V$, $1 \cdot v = v$, where 1 is

the multiplicative identity in $\mathbb{F}$.
9. Distributivity of scalar multiplication with respect to vector addition: For all $a \in \mathbb{F}$, $u, v \in V$, $a \cdot (u + v) = a \cdot u + a \cdot v$.
10. Distributivity of scalar multiplication with respect to field addition: For all $a, b \in \mathbb{F}$, $v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$.

A set with operations satisfying all these axioms is a vector space.

Assessing Sets and Operations for Vector Space Criteria

Let’s examine various examples, analyze whether they form vector spaces, and understand the reasons behind their classification.

Example 1: The Set of All Real Numbers with Usual Addition and Scalar Multiplication

Set: $V = \mathbb{R}$

Operations: Addition $+$ is standard real addition; scalar multiplication $\cdot$ is standard real multiplication.

Analysis:

The real numbers with standard addition and scalar multiplication form a vector space over $\mathbb{R}$. All axioms are satisfied because:

- Closure under addition and scalar multiplication holds.
- Additive identity is 0.
- Additive inverses exist (negatives).
- Scalar multiplication respects the field properties.

Conclusion: Yes, $\mathbb{R}$ with standard operations is a vector space.

Example 2: The Set of All Non-Negative Real Numbers $\mathbb{R}_{\geq 0}$ with Usual Addition and Scalar Multiplication

Set: $V = \mathbb{R}_{\geq 0}$

Operations: Usual addition and scalar multiplication.

Analysis:

- Closure under addition: Yes, sum of two non-negative numbers is non-negative.
- Closure under scalar multiplication: No, if scalar $a < 0$ and $v > 0$, then $a \cdot v < 0$, which is outside the set.
- Additive identity: 0 is in the set.
- Additive inverses: Not in the set for any $v > 0$ because $-v \notin \mathbb{R}_{\geq 0}$.

Violation: Closure under scalar multiplication fails when scalars are negative.

Conclusion: No, since the set is not closed under scalar multiplication with all field elements, it is not a vector space.

Example 3: The Set of All 2×2 Matrices with Real Entries

Set: $V = M_{2 \times 2}(\mathbb{R})$

Operations: Matrix addition and scalar multiplication as usual.

Analysis:

- Closure under addition: Yes.
- Closure under scalar multiplication: Yes.
- Additive identity: Zero matrix.
- Additive inverses: Negative matrices.
- All other axioms hold.

Conclusion: Yes, $M_{2 \times 2}(\mathbb{R})$ with standard operations is a vector space.

Example 4: The Set of All Polynomials of Degree Exactly 3

Set: $V = \{ p(x) = ax^3 + bx^2 + cx + d \mid a \neq 0 \}$

Operations: Polynomial addition and scalar multiplication.

Analysis:

- Closure under addition: No, sum of two degree 3 polynomials may be degree 3 if leading

coefficients sum to zero, possibly resulting in a polynomial of degree less than 3.
- Scalar multiplication: If scalar $a=0$, then $a p(x)$ is the zero polynomial, which has degree less than 3.

Violation: Closure fails because the sum or scalar multiple may leave the set of degree exactly 3.

Conclusion: No, this set is not closed under addition or scalar multiplication; thus, it is not a vector space.

Example 5: The Set of All Pairs (x, y) where $y = 2x$ in $\mathbb{R}^2$

Set: $V = \{ (x, y) \in \mathbb{R}^2 \mid y = 2x \}$

Operations: Usual vector addition and scalar multiplication.

Analysis:

- Closure under addition: Sum of two vectors on the line $y=2x$ remains on the line.
- Closure under scalar multiplication: Scalar multiplication of a vector on the line remains on the line.
- Contains zero vector: $(0,0)$ satisfies $y=2x$.

Conclusion: Yes, this set is a line through the origin, which is a subspace of $\mathbb{R}^2$.

Common Violations That Prevent Sets From Being Vector Spaces

When analyzing a set, certain violations of the axioms are common:

1. Lack of Closure Under Addition or Scalar Multiplication

A set must be closed under both operations for it to be a vector space. For example, the set of positive real numbers is not closed under scalar multiplication by negative scalars, violating the closure axiom.

2. Absence of Additive Identity or Inverses

If the set does not contain the zero vector or does not allow for additive inverses, it cannot be a vector space.

3. Inconsistency in the Set's Definition

Sets defined by strict degree conditions or other properties that are not preserved under addition or scalar multiplication typically fail to be vector spaces.

Special Cases and Subspaces

Some sets are subspaces of larger vector spaces, which automatically satisfy the axioms inherited from the parent space. For example, the set of all polynomials of degree less than or equal to n forms a subspace of the polynomial space.

Criteria for Subspaces

A subset $W \subseteq V$ is a subspace if:

- It contains the zero vector.
- It is closed under addition.
- It is closed under scalar multiplication.

If these conditions hold, the subset is a vector space in its own right

Frequently Asked Questions

What are the key properties to verify when determining if a set with given operations forms a vector space?

You need to check if the set is closed under vector addition and scalar multiplication, and verify the existence of an additive identity and additive inverses, as well as the distributive, associative, and scalar multiplication properties as per vector space axioms.

How do I determine if the set of all polynomials of degree less than or equal to 3, with usual addition and scalar multiplication, is a vector space?

This set is a vector space because it is closed under addition and scalar multiplication, contains the zero polynomial, and satisfies all vector space axioms with the usual operations.

Can a set of non-zero vectors with an operation that does not satisfy closure be considered a vector space?

No, closure under addition and scalar multiplication is fundamental; if this property fails, the set cannot be a vector space.

Is the set of all real-valued functions on an interval with pointwise addition and scalar multiplication a vector space?

Yes, because the set is closed under addition and scalar multiplication, and all vector space axioms are satisfied with these operations.

If a set is equipped with an operation that is not associative, can it still be a vector space?

No, associativity of addition is a core axiom of vector spaces; without it, the structure cannot be a vector space.

How do I analyze whether the set of invertible matrices forms a vector space under matrix addition?

Since the sum of invertible matrices may not be invertible, the set is not closed under addition and thus does not form a vector space.

What should I conclude if a set of vectors does not have an additive identity within the set?

The set cannot be a vector space because the existence of an additive identity (zero vector) within the set is a necessary axiom.

Can the set of all solutions to a homogeneous linear differential equation form a vector space?

Yes, the set of solutions to a homogeneous linear differential equation is closed under addition and scalar multiplication, satisfying all vector space axioms.

If a set with given operations fails to satisfy one of the vector space axioms, what is the implication?

The set is not a vector space; identifying which axiom fails helps determine the specific reason and guides possible modifications.

Additional Resources

Determining Whether a Set Equipped with Given Operations Constitutes a Vector Space: An In-Depth Analysis

In the realm of linear algebra, the concept of a vector space forms a foundational pillar that underpins many areas of mathematics, engineering, physics, and computer science. Establishing whether a particular set, endowed with specified operations, qualifies as a vector space is essential for both theoretical understanding and practical applications. This analytical process involves examining the set's structure relative to the defining axioms of vector spaces, which include properties such as closure, associativity, identity elements, inverses, and distributivity. This article provides a comprehensive exploration of how to determine whether given sets with particular operations are vector spaces, with detailed explanations, illustrative examples, and critical insights into cases where they are not. We aim to equip readers with the tools and understanding necessary to evaluate such structures rigorously and appreciate the nuances involved in their classification.

Understanding the Foundations of Vector Spaces

What Is a Vector Space?

At its core, a vector space over a field $\mathbb{F}$ (commonly $\mathbb{R}$ or $\mathbb{C}$) is a set V equipped with two operations: vector addition and scalar multiplication. These operations must satisfy eight fundamental axioms, ensuring the structure behaves consistently with the intuitive notions of vectors in Euclidean space.

Formally, a set V is a vector space over $\mathbb{F}$ if:

1. Closure under addition: For all $u, v \in V$, $u + v \in V$.
2. Associativity of addition: For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
3. Existence of additive identity: There exists an element $0 \in V$ such that for all $v \in V$, $v + 0 = v$.
4. Existence of additive inverses: For each $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
5. Closure under scalar multiplication: For all $a \in \mathbb{F}$ and $v \in V$, $a v \in V$.
6. Distributivity of scalar over vector addition: For all $a \in \mathbb{F}$, $u, v \in V$, $a (u + v) = a u + a v$.
7. Distributivity of scalar over field addition: For all $a, b \in \mathbb{F}$, $v \in V$, $(a + b) v = a v + b v$.
8. Compatibility of scalar multiplication with field multiplication: For all $a, b \in \mathbb{F}$, $v \in V$, $(ab) v = a (b v)$.

These axioms collectively ensure that the set behaves as a linear structure, enabling concepts like linear combinations, bases, and dimensions.

General Methodology for Verifying Vector Space Properties

Determining whether a set with given operations is a vector space involves systematically verifying the axioms. The typical approach encompasses:

- Examining the set's closure properties: Are the operations closed within the set?
- Identifying the additive identity and inverses: Does the set contain an additive zero? Can each element be "negated" within the set?
- Checking scalar multiplication compatibility: Are the scalar multiplication rules consistent with the field's properties?
- Validating distributive and associative properties: Do the operations satisfy distributivity and associativity?

If any axiom fails, the set cannot be a vector space. Conversely, if all axioms hold, the set qualifies as a vector space.

Cases Where Sets Are Vector Spaces

Before delving into the exceptions, it's instructive to recognize typical instances where sets are known vector spaces.

Standard Vector Spaces

- Euclidean space \(\mathbb{R}^n \): With usual vector addition and scalar multiplication.
- Function spaces: Sets like \(C[a, b] \), the set of continuous functions on an interval, with pointwise addition and scalar multiplication.
- Polynomial spaces: The set of all polynomials of degree less than or equal to \(n \).

In these classic cases, the operations clearly satisfy all axioms, and the structure aligns with the definition of a vector space.

Analyzing Sets with Given Operations: When Are

They Not Vector Spaces?

While many familiar sets are vector spaces, numerous interesting structures fail to meet the criteria due to subtle or overt violations of the axioms. Below, we explore common scenarios and detailed examples.

Scenario 1: Operations Are Not Closed Within the Set

Example: Consider the set $V = \{ (x, y) \in \mathbb{R}^2 \mid y = \frac{1}{x} \text{ for } x \neq 0 \}$, with the standard addition and scalar multiplication.

Analysis:

- Closure under addition: For $(x_1, \frac{1}{x_1})$ and $(x_2, \frac{1}{x_2})$, their sum is $(x_1 + x_2, \frac{1}{x_1} + \frac{1}{x_2})$.

- Is the sum in V? For the sum to be in V, the second component must be $1/(x_1 + x_2)$. But:

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_2 + x_1}{x_1 x_2} \neq \frac{1}{x_1 + x_2}$$

unless special cases occur, which generally do not hold. Therefore, the set is not closed under addition.

Conclusion: Since closure fails, V is not a vector space.

Scenario 2: The Zero Vector Does Not Exist in the Set

Example: Take the set $W = \{ (x, y) \in \mathbb{R}^2 \mid y = 1 \}$ with the usual addition and scalar multiplication.

Analysis:

- Zero vector: In $\mathbb{R}^2$, the zero vector is $(0, 0)$. Is $(0, 0) \in W$? No, because $y = 1$ for all elements in W, so $(0,0) \notin W$.

- Implication: Without the additive identity, the set cannot be a vector space.

Conclusion: The absence of the zero vector disqualifies W from being a vector space.

Scenario 3: The Set Is Not Closed Under Scalar Multiplication

Example: Consider the set $U = \{ (x, y) \in \mathbb{R}^2 \mid y = x^2 \}$ with standard addition but modified scalar multiplication defined as:

$$ a \ast (x, y) = (a x, a^2 y) $$

Analysis:

- Check closure under the new scalar multiplication: For $(x, y) \in U$, $y = x^2$. Applying scalar a:

$$ a \ast (x, y) = (a x, a^2 y) $$

- Is the result in U? To verify whether the new vector is in U:

$$ y' = (a x)^2 = a^2 x^2 $$

- Compare with: $a^2 y = a^2 x^2$.

Since $y' = a^2 y$, the new vector satisfies $y' = (a x)^2$, which aligns with the set's defining relation.

However, the catch is in the scalar multiplication's definition: if this operation is not compatible with the field's properties (e.g., scalar multiplication is non-standard or does not satisfy distributivity), then the structure may fail to be a vector space.

Suppose scalar multiplication is defined differently, say:

$$ a \ast (x, y) = (a x, a y) \quad \text{if } a \neq 0, \text{ and } (0, 0) \text{ if } a=0 $$

and the set remains $y = x^2$.

- Check whether the scalar multiplication preserves the property: For $(x, y) \in U$, after scalar multiplication:

$$ (a x, a y) $$

- The new vector satisfies $y' = (a x)^2 = a^2 x^2$, but the second component is $a y$. To have the image in U, the relation must be:

\[a y = (a x)^2 \Right

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