Determine Whether The Following Infinite Series Possess A Finite Sum. If They Do, A Determine The Value

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Infinite series are fundamental objects in mathematical analysis, with numerous applications across science, engineering, economics, and beyond. Understanding whether an infinite series converges—i.e., possesses a finite sum—is crucial for both theoretical insights and practical computations. In this article, we explore the techniques used to analyze various types of infinite series, determine their convergence or divergence, and, where possible, find their exact sums.

Understanding Infinite Series

An infinite series is the sum of infinitely many terms, typically expressed as:

$$ S = \sum_{n=1}^{\infty} a_n $$

where a_n is the n-th term of the series. The core question is whether this sum converges to a finite value or diverges to infinity or does not settle to a limit at all.

Key Concepts and Definitions

Convergence and Divergence

- Convergent Series: An infinite series $\sum_{n=1}^\infty a_n$ converges if the sequence of partial sums

$$ S_N = \sum_{n=1}^N a_n $$

approaches a finite limit as $N \to \infty$.

- Divergent Series: If the sequence of partial sums does not approach a finite limit, the series diverges.

Tests for Convergence

Several tests help determine whether an infinite series converges or diverges:

- **nth-Term Test:** If $\lim_{n \to \infty} a_n \neq 0$, the series diverges.
- **Geometric Series Test:** For $\sum ar^{n}$, the series converges if $|r| < 1$ and diverges otherwise.
- **P-Series Test:** For $\sum \frac{1}{n^p}$, the series converges if $p > 1$, diverges if $p \leq 1$.
- **Comparison Test:** Compare with a known convergent or divergent series.
- **Limit Comparison Test:** Use when terms behave similarly to a known series.
- **Ratio Test:** Examine $\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}|$. If less than 1, series converges; if greater than 1, diverges.
- **Root Test:** Consider $\lim_{n \to \infty} \sqrt[n]{|a_n|}$. Similar conclusions as Ratio Test.

Analyzing Specific Types of Infinite Series

Different series types require different approaches for analysis. Here, we'll examine common series classes, their convergence criteria, and how to compute their sums when they converge.

Geometric Series

A geometric series takes the form:

$$\sum_{n=0}^\infty ar^n$$

where a is the first term, and r is the common ratio.

Convergence Criterion

- The series converges if and only if $|r| < 1$.

- The sum to infinity (if convergent) is:

$$ S = \frac{a}{1 - r} $$

Example

Evaluate:

$$ \sum_{n=0}^\infty \left(\frac{1}{2}\right)^n $$

Since $a = 1$ and $r = 1/2$,

$$ S = \frac{1}{1 - 1/2} = 2 $$

P-Series

Series of the form:

$$ \sum_{n=1}^\infty \frac{1}{n^p} $$

are called p-series.

Convergence Criterion

- For $p > 1$, the series converges.
- For $p \leq 1$, the series diverges.

Examples

- $\sum_{n=1}^\infty \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$.
- $\sum_{n=1}^\infty \frac{1}{n}$ (harmonic series) diverges.

Alternating Series

Series where signs alternate, e.g.,

$$ \sum_{n=1}^\infty (-1)^{n+1} a_n $$

with $a_n > 0$.

Alternating Series Test (Leibniz Test)

- If a_n is decreasing and $\lim_{n \to \infty} a_n = 0$, then the

series converges.
- The sum can often be approximated, but finding an exact sum is more challenging.

Power Series

Series of the form:

$$ \sum_{n=0}^\infty c_n (x - a)^n $$

which converge within a radius known as the radius of convergence. These are fundamental in representing functions.

Determining Whether a Series Has a Finite Sum: Step-by-Step Approach

When analyzing a specific series, follow these steps:

1. **Check the terms:** Compute $\lim_{n \to \infty} a_n$. If not zero, the series diverges.

2. **Identify the type of series:** Is it geometric, p-series, telescoping, alternating, or power series?

3. **Apply relevant convergence tests:** Use the appropriate test based on the series type.

4. **Calculate the sum if convergent:** Use known formulas or techniques such as partial fraction decomposition, known series sums, or generating functions.

Examples of Series Analysis

Example 1: Geometric Series

Evaluate:

$$ \sum_{n=0}^\infty 3 \left(\frac{1}{4}\right)^n $$

Solution:

- First term $a = 3$, ratio $r = 1/4$.
- $|r| < 1$, so series converges.
- Sum:

$$ S = \frac{a}{1 - r} = \frac{3}{1 - 1/4} = \frac{3}{3/4} = 4 $$

Conclusion: The series converges to 4.

Example 2: p-Series

Evaluate:

$$ \sum_{n=1}^\infty \frac{1}{n^3} $$

Solution:

- Since $p = 3 > 1$, the series converges.
- Its sum is known as the Riemann zeta function at 3:

$$ \zeta(3) \approx 1.2020569 $$

Conclusion: The series converges to approximately 1.2021.

Example 3: Alternating Series

Evaluate:

$$ \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} $$

Solution:

- $a_n = 1/n$, decreasing and tending to 0.
- Series converges by the Alternating Series Test.
- The sum is known as the alternating harmonic series, which converges to $\ln 2$.

Conclusion: The series converges to $\ln 2 \approx 0.6931$.

When Series Do Not Possess a Finite Sum

Understanding divergence is equally important. Series that do not meet the convergence criteria do not have finite sums. For example:

- The harmonic series diverges.
- The geometric series with $|r| \geq 1$ diverges.
- Series where the terms do not tend to zero diverge, such as $\sum_{n=1}^\infty 1$.

Special Techniques for Complex Series

Some series require advanced methods:

Telescoping Series

Series where terms cancel out, enabling straightforward summation.

Example:

$$\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Sum:

$$S_N = 1 - \frac{1}{N+1} \to 1 \quad \text{as } N \to \infty$$

Conclusion: The series converges to 1.

Series Using Partial Fraction Decomposition

Decompose complex rational functions into simpler fractions to analyze sums.

Conclusion

Determining whether an infinite series possesses a finite sum involves a systematic approach: examining the behavior of terms, applying appropriate convergence tests, and leveraging known sum

Frequently Asked Questions

Does the infinite geometric series 3 + 1.5 + 0.75 + ... converge, and if so, what is its sum?

Yes, it converges because the common ratio $r = 0.5 < 1$. The sum is $S = a / (1 - r) = 3 / (1 - 0.5) = 3 / 0.5 = 6$.

Determine whether the series $\sum_{n=1}^\infty 1/n^2$ converges, and find its sum if it does.

Yes, the series converges because it is a p-series with $p = 2 > 1$. Its sum is known as $\zeta(2)$, which equals $\pi^2 / 6 \approx 1.6449$.

Is the harmonic series $\sum_{n=1}^\infty 1/n$ convergent or divergent? If convergent, what is its sum?

The harmonic series diverges; it does not have a finite sum.

Evaluate the convergence and sum of the series $\sum_{n=1}^\infty (-1)^{n+1} / n$.

Yes, the alternating harmonic series converges by the Alternating Series Test, and its sum is $\ln(2)$.

Determine if the series $\sum_{n=1}^\infty (2^n) / (3^n)$ converges, and find its sum if it does.

Yes, it converges because the ratio $r = (2/3) < 1$. The sum is $S = a / (1 - r) = (2/3) / (1 - 2/3) = (2/3) / (1/3) = 2$.

Does the series $\sum_{n=1}^\infty n / 2^n$ have a finite sum? If so, what is it?

Yes, it converges because it is a power series with $|r| < 1$. The sum is $S = r / (1 - r)^2$, where $r = 1/2$, so $S = (1/2) / (1 - 1/2)^2 = (1/2) / (1/2)^2 = (1/2) / (1/4) = 2$.

Analyze the series $\sum_{n=1}^\infty n! / n^n$ for convergence and determine its sum if finite.

The series converges because $n! / n^n$ tends to 0 very rapidly as n approaches infinity. However, it does not have a simple closed-form sum.

Additional Resources

Infinite Series Analysis: Determining Convergence and Exact Sums

When exploring the fascinating realm of infinite series in mathematics, one of the most compelling questions is whether a given series converges—that is, whether it approaches a finite value as the number of terms increases indefinitely—and, if so, what that value might be. This inquiry not only deepens our understanding of mathematical structures but also has practical implications across physics, engineering, computer science, and many other fields. In this article, we will delve into the methodologies used to analyze infinite series, demonstrate how to determine whether they possess finite sums, and, where possible, compute those sums with precision.

Understanding Infinite Series: The Foundations

Before examining specific series, it's essential to grasp the core concepts.

What Is an Infinite Series?

An infinite series is the sum of infinitely many terms, typically expressed as:

$$
S = \sum_{n=1}^\infty a_n
$$

where a_n is the general term of the series.

Unlike finite sums, infinite series can either converge to a finite limit or diverge to infinity or oscillate without settling on a specific value.

Convergence vs. Divergence

- Convergent Series: The series approaches a finite limit, called the sum. That is, the sequence of partial sums:

$$
S_N = \sum_{n=1}^N a_n
$$

approaches a specific number S as $N \to \infty$.

- Divergent Series: The partial sums do not approach a finite limit. They may grow without bound, oscillate, or fail to settle.

Understanding whether a series converges is the first step in ascertaining if it has a finite sum.

Methods to Determine Convergence and Sum

To analyze whether a series has a finite sum, mathematicians use several tests and techniques. The choice of method depends on the form of the series.

1. The Nth Term Test for Divergence

Purpose: Quickly determine if a series diverges.

Procedure: If $\lim_{n \to \infty} a_n \neq 0$, then the series diverges.

Note: If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive.

Example: For $\sum_{n=1}^\infty \frac{1}{n}$, since $\lim_{n \to \infty} \frac{1}{n} = 0$, the test is inconclusive; further tests are needed.

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2. Geometric Series Test

Series form: $\sum_{n=0}^\infty ar^n$

Convergence criterion: If $|r| < 1$, the series converges.

Sum formula:

$$
S = \frac{a}{1 - r}
$$

Implication: Many series can be identified as geometric, allowing for straightforward convergence checks and sum calculations.

Example: $\sum_{n=0}^\infty \left(\frac{1}{2}\right)^n$

Since $|1/2| < 1$, the series converges to $1/(1 - 1/2) = 2$.

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3. p-Series Test

Series form: $\sum_{n=1}^\infty \frac{1}{n^p}$

Convergence criterion: The series converges if and only if $p > 1$.

Examples:

- $\sum_{n=1}^\infty \frac{1}{n^2}$ converges (the Basel problem, sum equals $\pi^2/6$).

- $\sum_{n=1}^\infty \frac{1}{n}$ diverges (harmonic series).

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4. Comparison and Limit Comparison Tests

Comparison Test: If $0 \leq a_n \leq b_n$ for large n, and $\sum b_n$ converges, then $\sum a_n$ converges; if $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit Comparison Test: If

$$
\lim_{n \to \infty} \frac{a_n}{b_n} = c
$$

where c is a finite positive number, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

---

5. Integral Test

Used when $a_n = f(n)$ with f positive, decreasing, continuous.

Procedure: Evaluate

$$
\int_1^\infty f(x) \, dx
$$

If the integral converges, so does the series; if it diverges, so does the series.

Example: For $a_n = 1/n^p$, integral test confirms convergence if $p > 1$.

Analyzing Specific Series: Step-by-Step Approach

Suppose we are presented with a list of series, and our task is to determine whether they possess finite sums, and if so, compute those sums.

Series 1: The Geometric Series $\sum_{n=0}^\infty \left(\frac{1}{3}\right)^n$

- Analysis: Recognized as a geometric series with $a = 1$, $r = 1/3$.

- Convergence: Since $|r| = 1/3 < 1$, the series converges.

- Sum Calculation:

$$
S = \frac{a}{1 - r} = \frac{1}{1 - 1/3} = \frac{1}{2/3} = \frac{3}{2}
$$

Verdict: Finite sum, value $\boxed{\frac{3}{2}}$.

Series 2: The p-Series $\sum_{n=1}^\infty \frac{1}{n^2}$

- Analysis: Recognized as a p-series with $p=2 > 1$.

- Convergence: Series converges.

- Sum: Known as the Basel problem, the sum is:

$$
\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449
$$

Verdict: Finite sum, value $\boxed{\frac{\pi^2}{6}}$.

Series 3: Alternating Harmonic Series \(\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \)

- Analysis: This is an alternating series with decreasing terms tending to zero.

- Test Applied: Alternating Series Test (Leibniz Criterion):

- \(a_n = 1/n \), which decreases monotonically to zero.

- Conclusion: Series converges (conditionally).

- Sum: Known as the Leibniz series, sum equals \(\ln(2) \).

Verdict: Finite sum, value \(\boxed{\ln(2)} \).

Series 4: The Harmonic Series \(\sum_{n=1}^\infty \frac{1}{n} \)

- Analysis: p-series with \(p=1 \).

- Convergence: Diverges (the harmonic series is well-known to diverge).

Verdict: No finite sum.

Series 5: The Series \(\sum_{n=1}^\infty \frac{n}{2^n} \)

- Analysis: Recognize as a weighted geometric series.

- Method: Use the formula for the sum of \(\sum_{n=1}^\infty n r^{n} \):

\[
\sum_{n=1}^\infty n r^{n} = \frac{r}{(1 - r)^2}
\]

for \(|r| < 1 \).

- Calculation:

\[

r = \frac{1}{2} \Rightarrow S = \frac{\frac{1}{2}}{(1 - \frac{1}{2})^2} =
\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} =
\frac{\frac{1}{2}}{\frac{1}{4}} = 2
\]

Verdict: Finite sum, value \(\boxed{2} \).

Advanced Techniques for Complex Series

While the above methods cover many common series, some series require more sophisticated techniques:

- Power Series Expansion: Using known power series (e.g., exponential, sine, cosine) to identify sums.

- Generating Functions: For sequences with combinatorial significance.

- Fourier Series: To analyze series involving trigonometric functions.

- Contour Integration

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