Evaluate C (X 2 Y 2) D S C(x2 Y2)ds , C Is The Top Half Of The Circle With Radius 6 Centered At (0,0)

Evaluate C (X 2 Y 2) D S C(x2 Y2)ds , C Is The Top Half Of The Circle With Radius 6 Centered At (0,0)

Introduction

When studying line integrals in multivariable calculus, a common problem involves evaluating the integral of a scalar function over a specified curve. In this case, we are asked to evaluate the line integral of the function $f(x, y) = x^2 y^2$ over the curve C, which is the top half of a circle with radius 6 centered at the origin $(0, 0)$. This problem combines several key concepts of calculus, including parametrization of curves, setting up integrals in terms of parameters, and applying symmetry to simplify calculations.

Understanding how to approach such integrals is essential for students and professionals working in fields like physics, engineering, and mathematics, where line integrals frequently appear in problems related to work, flux, and field evaluations. In this article, we will explore step-by-step how to evaluate the integral $\int_C x^2 y^2 ds$, where C is the specified semicircular arc.

Defining the Curve C

The Geometry of the Curve

The curve C is described as the top half of a circle with:

- Radius $r = 6$
- Center at $(0, 0)$

This circle can be represented mathematically as:

$$
x^2 + y^2 = 36
$$

Since C is only the top half of this circle, the y-coordinate will be non-negative:

$$
y \geq 0
$$

Parametrization of C

To evaluate the line integral, we need to parametrize the curve C. A standard parametrization of a circle of radius r centered at the origin is:

$$
x(t) = r \cos t, \quad y(t) = r \sin t
$$

where t ranges over an interval corresponding to the portion of the circle we are considering.

For the top half of the circle:

- t varies from 0 to π
- The parametrization becomes:

$$
x(t) = 6 \cos t, \quad y(t) = 6 \sin t, \quad t \in [0, \pi]
$$

Derivatives of the Parametric Equations

To compute ds, the differential arc length, we need the derivatives:

$$
\frac{dx}{dt} = -6 \sin t, \quad \frac{dy}{dt} = 6 \cos t
$$

The differential arc length element is:

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt
$$

which simplifies to:

$$
ds = \sqrt{36 \sin^2 t + 36 \cos^2 t} dt = \sqrt{36 (\sin^2 t + \cos^2 t)} dt = \sqrt{36 \times 1} dt = 6 dt
$$

Setting Up the Line Integral

Expressing Integrand in Terms of t

The integral we need to evaluate is:

$$
\int_C x^2 y^2 ds
$$

Substituting the parametrizations:

$$
x(t) = 6 \cos t, \quad y(t) = 6 \sin t
$$

the integrand becomes:

$$
x^2 y^2 = (6 \cos t)^2 (6 \sin t)^2 = 36 \cos^2 t \times 36 \sin^2 t = 1296 \cos^2 t \sin^2 t
$$

Expressing ds in Terms of t

From previous calculations:

$$
ds = 6 dt
$$

The Integral in Terms of t

Putting everything together:

$$
\int_{t=0}^{\pi} (1296 \cos^2 t \sin^2 t) \times 6 dt = 1296 \times 6 \int_{0}^{\pi} \cos^2 t \sin^2 t dt
$$

which simplifies to:

$$
7776 \int_{0}^{\pi} \cos^2 t \sin^2 t dt
$$

Simplifying the Integral

Using Trigonometric Identities

The integral:

$$
I = \int_{0}^{\pi} \cos^2 t \sin^2 t dt
$$

can be simplified using identities:

$$
\sin^2 t = \frac{1 - \cos 2t}{2}
$$

$$
\cos^2 t = \frac{1 + \cos 2t}{2}
$$

Thus,

$$
I = \int_{0}^{\pi} \left(\frac{1 + \cos 2t}{2}\right) \left(\frac{1 - \cos 2t}{2}\right) dt
$$

Multiplying the numerators:

$$
I = \frac{1}{4} \int_{0}^{\pi} (1 + \cos 2t)(1 - \cos 2t) dt
$$

Expanding:

$$
(1 + \cos 2t)(1 - \cos 2t) = 1 - \cos^2 2t
$$

Therefore,

$$
I = \frac{1}{4} \int_{0}^{\pi} (1 - \cos^2 2t) dt
$$

Evaluating the Integral

Split the integral:

$$
I = \frac{1}{4} \left(\int_{0}^{\pi} 1 dt - \int_{0}^{\pi} \cos^2 2t dt \right)
$$

The first integral:

$$
\int_{0}^{\pi} 1 dt = \pi
$$

The second integral involves $\cos^2 2t$:

$$
\int_{0}^{\pi} \cos^2 2t dt
$$

Using the identity:

$$
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
$$

set $\theta = 2t$:

$$
\cos^2 2t = \frac{1 + \cos 4t}{2}
$$

Thus:

$$
\int_{0}^{\pi} \cos^2 2t dt = \frac{1}{2} \int_{0}^{\pi} (1 + \cos 4t) dt
$$

which equals:

$$
\frac{1}{2} \left(\int_{0}^{\pi} 1 dt + \int_{0}^{\pi} \cos 4t dt \right)
$$

Compute each:

- $\int_{0}^{\pi} 1 dt = \pi$
- $\int_{0}^{\pi} \cos 4t dt = \frac{\sin 4t}{4} \bigg|_{0}^{\pi} = \frac{\sin 4\pi - \sin 0}{4} = 0$

Therefore:

$$
\int_{0}^{\pi} \cos^2 2t dt = \frac{1}{2} (\pi + 0) = \frac{\pi}{2}
$$

Final Calculation of I

Putting it all together:

$$
I = \frac{1}{4} \left(\pi - \frac{\pi}{2} \right) = \frac{1}{4} \times \frac{\pi}{2} = \frac{\pi}{8}
$$

Final Result of the Line Integral

Recall that the original integral is:

$$
\int_C x^2 y^2 ds = 7776 \times I = 7776 \times \frac{\pi}{8}
$$

Simplify:

$$
\frac{7776}{8} = 972
$$

Thus, the value of the line integral is:

$$
\boxed{
\int_C x^2 y^2 ds = 972 \pi
}
$$

Summary and Key Takeaways

- The curve C is parametrized as $x(t) = 6 \cos t$, $y(t) = 6 \sin t$, with $t \in [0, \pi]$.
- The differential arc length ds simplifies to $6 dt$.
- The integrand $x^2 y^2$ becomes $1296 \cos^2 t \sin^2 t$.
- The integral reduces to evaluating $\int_0^{\pi} \cos^2 t \sin^2 t dt$, which simplifies via trigonometric identities.
- The final answer for the line integral over the top half of the circle of radius 6 is $\boxed{972 \pi}$.

Additional Insights

Symmetry and Simplification

Using symmetry and trigonometric identities significantly simplifies the computation of such integrals. Recognizing that $\cos^2 t \sin^2 t$ can

Frequently Asked Questions

What is the geometric shape of the curve C in the given problem?

The curve C is the top half of a circle with radius 6 centered at the origin (0,0).

How is the curve C parameterized for evaluating the line integral?

Since C is the top half of a circle of radius 6, it can be parameterized as $x = 6 \cos t$, $y = 6 \sin t$, with t ranging from 0 to π.

What is the form of the vector field involved in the line integral?

The integrand C(x^2 y^2) D S C(x^2 y^2) ds suggests a scalar or vector field involving x^2 y^2; clarification is needed, but typically, it could involve a scalar function multiplied by a differential element.

How do you compute the differential arc length ds along the curve C?

ds can be computed using the derivatives dx/dt and dy/dt: ds = sqrt((dx/dt)^2 + (dy/dt)^2) dt.

What are the limits of integration for the parameter t in this problem?

Since C is the top half of the circle, t varies from 0 to π.

How do you evaluate the line integral over the curve C?

Substitute the parameterization into the integrand, compute ds, and integrate with respect to t from 0 to π.

Are there symmetry properties that can simplify the evaluation of the integral?

Yes, the symmetry of the circle and the integrand (if it involves even powers) may simplify the calculation, especially for symmetric limits.

What potential challenges might arise when computing this line integral?

Challenges include correctly parameterizing the curve, computing derivatives accurately, and evaluating the integral of potentially complex functions involving x^2 y^2.

How does the radius of 6 influence the evaluation of the line integral?

The radius determines the scale of the circle and influences the values of x and y in the parameterization, affecting the integrand's magnitude and the differential elements.

Additional Resources

Evaluation of the Line Integral $\int_{C} (x^2 y^2) \, ds$: An In-depth Analysis of a Semicircular Path

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Introduction

In the realm of vector calculus and multivariable analysis, line integrals serve as fundamental tools for understanding the behavior of scalar and vector fields along specified paths. The integral $\int_{C} x^2 y^2 \, ds$ exemplifies a scalar line integral where the integrand is a function of the coordinates, and the path C is a geometric curve—in this case, the top half of a circle with radius 6 centered at the origin. This particular problem combines geometric intuition with analytical computation, inviting a detailed exploration of the methodology, underlying concepts, and potential implications.

This article aims to thoroughly dissect the evaluation process of this integral, starting from the geometric description of the path to the parameterization techniques, and culminating in the explicit calculation of the integral. Such an analysis not only clarifies the specific problem but also reinforces key principles in multivariable calculus relevant to similar applications across physics, engineering, and mathematics.

Geometric Description of the Path C

The Path C: The Top Half of a Circle of Radius 6

The curve C is specified as the top half of a circle with radius 6, centered at the origin $(0,0)$. Geometrically, this is a semicircular arc extending from $-6, 0$ to $6, 0$ along the upper boundary of the circle $x^2 + y^2 = 36$.

Key features:

- Center: $(0, 0)$
- Radius: $r = 6$
- Curve segment: Top semicircle, $y \geq 0$
- Endpoints: $(-6, 0)$ to $(6, 0)$

This geometric setup provides a symmetric domain that simplifies the integral's evaluation, thanks to the symmetry properties of the integrand and the path.

Mathematical Formulation of the Line Integral

The line integral in question:

$$
\int_{C} x^2 y^2 \, ds
$$

involves integrating the scalar function $f(x, y) = x^2 y^2$ along the curve C, with ds representing an infinitesimal arc length element.

Significance of the Integral

- Scalar field: The integrand measures a quantity depending on the position within the plane.
- Arc length differential: ds accounts for the variable speed along the path; it ensures the integral sums the field's value over the actual length of the curve, not just the projection.

Parameterization of the Curve C

Choosing an Appropriate Parameterization

To evaluate the integral, the key step involves expressing both the coordinates (x, y) and the differential ds in terms of a suitable parameter, typically an angle θ.

Given the circle $x^2 + y^2 = 36$, the natural parameterization uses the angle θ:

$$
x(\theta) = r \cos \theta, \quad y(\theta) = r \sin \theta
$$

with $r = 6$.

Since C is the top half of the circle:

- θ ranges from π to 0 (or equivalently from 0 to π), but to match the standard orientation (counterclockwise from the rightmost point), we set:

$$
\theta \in [0, \pi]
$$

Parameterization:

$$
x(\theta) = 6 \cos \theta, \quad y(\theta) = 6 \sin \theta, \quad \theta \in [0, \pi]
$$

\]

Computing \(ds\): The Differential Arc Length

The differential element \(ds\) along the curve is given by:

\[
ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 +
\left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]

Calculating derivatives:

\[
\frac{dx}{d\theta} = -6 \sin \theta, \quad \frac{dy}{d\theta} = 6 \cos \theta
\]

Thus:

\[
ds = \sqrt{(-6 \sin \theta)^2 + (6 \cos \theta)^2} \, d\theta = \sqrt{36
\sin^2 \theta + 36 \cos^2 \theta} \, d\theta
\]

Using the Pythagorean identity:

\[
\sin^2 \theta + \cos^2 \theta = 1
\]

we have:

\[
ds = \sqrt{36 (\sin^2 \theta + \cos^2 \theta)} \, d\theta = \sqrt{36 \times
1} \, d\theta = 6 \, d\theta
\]

This simplification indicates that the arc length element is constant along
the circle:

\[
ds = 6 \, d\theta
\]

Expressing the Integrand in Terms of \(\theta\)

Substituting \(x(\theta)\) and \(y(\theta)\):

$$
x(\theta) = 6 \cos \theta
$$
$$
y(\theta) = 6 \sin \theta
$$

The integrand becomes:

$$
x^2 y^2 = (6 \cos \theta)^2 \times (6 \sin \theta)^2 = 36 \cos^2 \theta \times 36 \sin^2 \theta = 1296 \cos^2 \theta \sin^2 \theta
$$

Setting Up the Integral in θ

Putting it all together:

$$
\int_{C} x^2 y^2 \, ds = \int_{0}^{\pi} 1296 \cos^2 \theta \sin^2 \theta \times 6 \, d\theta
$$

Simplify:

$$
= 1296 \times 6 \int_{0}^{\pi} \cos^2 \theta \sin^2 \theta \, d\theta
$$
$$
= 7776 \int_{0}^{\pi} \cos^2 \theta \sin^2 \theta \, d\theta
$$

Evaluating the Integral $\int_{0}^{\pi} \cos^2 \theta \sin^2 \theta \, d\theta$

This integral involves powers of sine and cosine functions, which are well-understood in calculus, especially with the use of power-reduction formulas.

Using Double-Angle Identities

Recall:

$$
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
$$
$$
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
$$

Therefore:

$$
\cos^2 \theta \sin^2 \theta = \left(\frac{1 + \cos 2\theta}{2}\right) \left(\frac{1 - \cos 2\theta}{2}\right) = \frac{(1 + \cos 2\theta)(1 - \cos 2\theta)}{4}
$$

Recognize the numerator as a difference of squares:

$$
(1 + \cos 2\theta)(1 - \cos 2\theta) = 1 - \cos^2 2\theta
$$

So:

$$
\cos^2 \theta \sin^2 \theta = \frac{1 - \cos^2 2\theta}{4}
$$

Now, the integral becomes:

$$
\int_{0}^{\pi} \cos^2 \theta \sin^2 \theta \, d\theta = \frac{1}{4} \int_{0}^{\pi} (1 - \cos^2 2\theta) \, d\theta
$$

Computing the Integral

Break down into two parts:

$$
\frac{1}{4} \left[\int_{0}^{\pi} 1 \, d\theta - \int_{0}^{\pi} \cos^2 2\theta \, d\theta \right]
$$

The first integral:

$$
\int_{0}^{\pi} 1 \, d\theta = \pi
$$

The second integral involves $\cos^2 2\theta$, which again can be simplified using the power-reduction formula:

$$
\cos^2 x = \frac{1 + \cos 2x}{2}
$$

```
\]

Applying this:

\[
\int_{0}^{\pi} \cos^2 2\theta \, d\theta = \int_{0}^{\pi} \frac{1 + \cos
4\theta}{2} \, d\theta = \frac{1}{2} \int_{0}^{\pi} 1 \, d\theta +
\frac{1}{2} \int_{0}^{\pi} \cos 4\theta
```

Evaluate C X 2 Y 2 D S C X2 Y2 Ds C Is The Top Half Of The Circle With Radius 6 Centered At 0 0

Evaluate C X 2 Y 2 D S C X2 Y2 Ds C Is The Top Half Of The Circle With Radius 6 Centered At 0 0

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