Find A Basis For And The Dimension Of The Solution Space Of The Homogeneous System Of Linear Equations.

Find A Basis For And The Dimension Of The Solution Space Of The Homogeneous System Of Linear Equations. Understanding how to determine a basis and the dimension of the solution space for a homogeneous system of linear equations is fundamental in linear algebra. These concepts are crucial for analyzing the structure of solutions, understanding vector spaces, and solving systems efficiently. In this comprehensive guide, we will explore what homogeneous systems are, how to find their solution spaces, and the methods to identify bases and dimensions, supported by detailed explanations and examples.

Understanding Homogeneous Systems of Linear Equations

Definition of a Homogeneous System

A system of linear equations is called homogeneous if all the constant terms are zero. It can be written in the form:

$$ A\mathbf{x} = \mathbf{0} $$

where:
- A is an $m \times n$ matrix of coefficients.
- $\mathbf{x}$ is an $n \times 1$ column vector of variables.
- $\mathbf{0}$ is the $m \times 1$ zero vector.

For example:

$$
\begin{cases}
2x + 3y - z = 0 \\
-x + 4y + 2z = 0 \\
\end{cases}
$$

This is a homogeneous system because all equations equal zero.

Solution Space of Homogeneous Systems

The set of all solutions to a homogeneous system forms a vector space called the null space or kernel of the matrix A. This space contains the trivial solution (all variables zero) and possibly infinitely many other solutions depending on the rank of A.

Key points:
- The solution space is always a subspace of $\mathbb{R}^n$.
- Its dimension is called the nullity of A.

Finding the Solution Space

Row Reduction to Echelon Form

The primary step in analyzing the solution space is to reduce the coefficient matrix A to its row echelon form (REF) or reduced row echelon form (RREF). This process simplifies the system, making it easier to identify free variables and express solutions.

Steps:
1. Use row operations (swap, multiply, add) to simplify A.
2. Achieve a form where the leading coefficients (pivots) are clear.
3. Identify pivots and free variables.

Expressing Solutions

Once in RREF, the system can be written in parametric form:

- Assign parameters to free variables.
- Express pivot variables in terms of these parameters.
- Write the general solution as a linear combination of vectors.

Example:
Suppose the RREF matrix gives the following equations:

$$
x_1 + 2x_3 = 0 \\
x_2 - x_3 = 0
$$

with $x_3 = t$, a free parameter. Then,

$$
x_1 = -2t \\
x_2 = t \\
x_3 = t
$$

The solution vector:

$$
\mathbf{x} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}
$$

indicates the solution space is spanned by the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Find A Basis for the Solution Space

What Is a Basis?

A basis of a vector space is a set of vectors that are linearly independent and span the entire space. For the solution space of a homogeneous system, the basis consists of the minimal set of vectors that generate all solutions through linear combinations.

How to find a basis:
1. Perform row reduction to RREF.
2. Identify the free variables.
3. Write the solutions in parametric form.
4. The vectors associated with the free variables form the basis.

Algorithm to Find the Basis

1. Reduce the matrix A to RREF.
2. Identify the pivot columns — columns with leading 1s.
3. Identify free variables — variables corresponding to non-pivot columns.
4. Express the pivot variables in terms of free variables.
5. Construct basis vectors by setting each free variable to 1 (and others to 0) in turn, then solving for the pivot variables.

Example:
Suppose the RREF of matrix A yields the following system:

$$
x_1 + 2x_3 = 0 \\
x_2 - x_3 = 0
$$

with x_3 free.

Set $x_3 = 1$:
$$
x_1 = -2, \quad x_2 = 1, \quad x_3 = 1
$$
Corresponding vector:

$$
\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}
$$

Set $x_3 = 0$:
$$

$$
x_1 = 0, \quad x_2 = 0, \quad x_3 = 0
$$
which is the trivial solution.

Thus, the basis for the solution space is:

$$
\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}
$$

Determining the Dimension of the Solution Space

Nullity of the Matrix

The dimension of the solution space, also called the nullity of the matrix A, equals the number of free variables in the system. It can be computed using the Rank-Nullity Theorem:

$$
\text{rank}(A) + \text{nullity}(A) = n
$$

where:
- $\text{rank}(A)$ is the number of pivot columns.
- n is the number of variables.

Steps to find dimension:
1. Compute the rank of A after row reduction.
2. Count the number of free variables (non-pivot columns).
3. The nullity is the number of free variables.

Example:
Suppose A is a 3×4 matrix with rank 2. Then:

$$
\text{nullity} = 4 - 2 = 2
$$

So, the solution space has dimension 2, and its basis consists of two linearly independent vectors.

Summary of Key Points

- The dimension of the solution space of a homogeneous system is the number of free variables.
- It equals $n - \text{rank}(A)$.
- The basis vectors can be obtained by setting each free variable to 1 (one at a time) and solving for the pivot variables.

Practical Applications and Examples

Example 1: Homogeneous System with Unique Solution

Suppose:

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
$$

The system:

$$
x_1 = 0, \quad x_2 = 0, \quad x_3 = 0
$$

has only the trivial solution. The basis is the empty set, and the dimension is zero.

Example 2: Homogeneous System with Infinite Solutions

Suppose:

$$
A = \begin{bmatrix}
1 & 2 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
$$

The system:

$$
x_1 + 2x_2 - x_3 = 0
$$

with x_2, x_3 free.

Set $x_2 = s$, $x_3 = t$:

$$
x_1 = -2s + t
$$

Solution vectors:

$$
\mathbf{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

The basis vectors are:

$$
\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

and the solution space has dimension 2.

Summary and Final Remarks

- Homogeneous systems always contain at least the trivial solution.
- The set of solutions forms a vector space called the null space.
- To find a basis, reduce the coefficient matrix to RREF, identify free variables, and construct vectors accordingly.
- The dimension of the solution space (nullity) is the number of free variables, which can be calculated using the rank-nullity theorem.

Importance in Linear Algebra:
Understanding the basis and dimension of the solution space helps in analyzing linear transformations, solving systems efficiently, and understanding the structure of vector spaces. These concepts extend to

Frequently Asked Questions

How do I find a basis for the solution space of a homogeneous system of linear equations?

To find a basis, first write the system in matrix form and reduce it to row echelon form. Then, identify the free variables and express the leading variables in terms of these free variables. The set of vectors corresponding to the free variables forms a basis for the solution space.

What is the relationship between the dimension of the solution space and the rank of the coefficient matrix?

The dimension of the solution space (nullity) equals the number of variables minus the rank of the coefficient matrix, as stated by the Rank-Nullity Theorem: nullity = number of variables - rank.

How can I determine the dimension of the solution space

without solving the entire system?

You can determine the dimension by finding the rank of the coefficient matrix and subtracting it from the total number of variables. Specifically, dimension = number of variables - rank of the matrix.

What does the solution space of a homogeneous system look like if the system has infinitely many solutions?

If the system has infinitely many solutions, the solution space is a subspace of the vector space with dimension greater than zero. Its basis can be found from the free variables after row reduction, representing the directions in which solutions extend.

Can the solution space of a homogeneous system be empty? Why or why not?

No, the solution space of a homogeneous system always contains at least the trivial solution (the zero vector), so it is never empty. It is a subspace that includes the trivial solution and possibly infinitely many others.

Additional Resources

Solution Space of Homogeneous Systems of Linear Equations: Finding a Basis and Determining Its Dimension

When delving into the fascinating world of linear algebra, one of the foundational topics is understanding the solution set of systems of linear equations. Particularly, homogeneous systems—those where all equations are set equal to zero—serve as essential building blocks for more advanced concepts such as vector spaces, linear transformations, and eigenvalues. A critical aspect of analyzing these systems involves finding a basis for their solution space and determining its dimension. This article provides an in-depth exploration of the methods, theoretical underpinnings, and practical strategies used for these tasks, presented with the clarity and rigor you'd expect from an expert review.

Understanding Homogeneous Systems of Linear Equations

What is a Homogeneous System?

A homogeneous system of linear equations can be written in the form:

$$
A \mathbf{x} = \mathbf{0}
$$

where:

- A is an $m \times n$ matrix with real (or complex) entries.
- $\mathbf{x}$ is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$).
- $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$ (or $\mathbb{C}^m$).

The key characteristic here is that the system is homogeneous—meaning the right-hand side is the zero vector. This ensures that the solution set always contains at least the trivial solution $\mathbf{x} = \mathbf{0}$, but often includes infinitely many solutions.

Properties of Homogeneous Systems

Understanding the structure of solutions involves several fundamental properties:

- Solution Set as a Subspace: The set of all solutions forms a vector subspace of $\mathbb{R}^n$. This is because the solutions are closed under addition and scalar multiplication, satisfying the axioms of a vector space.
- Existence of Non-trivial Solutions: If the system has more unknowns than independent equations (i.e., $\text{rank}(A) < n$), then non-trivial solutions exist (solutions other than the zero vector).
- Relation to Rank and Nullity: The structure of the solution space is deeply connected to the rank of matrix A and its nullity, as described by the Rank-Nullity Theorem.

Finding a Basis for the Solution Space

The core task involves extracting a basis for the solution space. A basis provides the minimal set of vectors needed to generate the entire solution set through linear combinations. The process involves systematic steps, typically utilizing matrix operations like row reduction.

Step 1: Set Up the System

Begin with the homogeneous system $A \mathbf{x} = \mathbf{0}$. Write down the augmented matrix:

$$
\text{Augmented matrix: } [A | \mathbf{0}]
$$

Since the right side is zero, focus on the coefficient matrix A.

Step 2: Row Reduce to Row Echelon Form

Perform elementary row operations (row swaps, scaling, and addition/subtraction of rows) to bring A into row echelon form (REF) or reduced row echelon form (RREF). The purpose is to identify:

- Leading variables (pivot variables)
- Free variables (non-pivot variables)

This simplification makes it straightforward to express solutions parametrically.

Step 3: Identify Free and Pivot Variables

- Pivot variables: Correspond to columns with leading entries in RREF.
- Free variables: Correspond to columns without leading entries.

The general solution will express pivot variables in terms of free variables, which serve as parameters.

Step 4: Express the General Solution

Write each pivot variable as a linear combination of free variables. The solution set will then be expressed as:

$$
\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k
$$

where:

- c_i are arbitrary scalars.
- $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \}$ are vectors obtained by assigning each free variable to 1 in turn (and others to 0), then solving for the pivot variables.

Step 5: Extract the Basis Vectors

The vectors $\mathbf{v}_i$ generated in the previous step form a basis for the solution space. These vectors are linearly independent by construction and span the entire solution space.

Example: Finding a Basis and Dimension

Suppose we have the following homogeneous system:

$$
\begin{cases}
x_1 + 2x_2 - x_3 = 0 \\
3x_1 + 6x_2 - 3x_3 = 0 \\
2x_1 + 4x_2 - 2x_3 = 0 \\
\end{cases}
$$

Step 1: Write the matrix:

$$
A = \begin{bmatrix}
1 & 2 & -1 \\
3 & 6 & -3 \\
2 & 4 & -2 \\
\end{bmatrix}
$$

Step 2: Row reduce to RREF:

- Subtract 3 times row 1 from row 2:

$$
R_2 \leftarrow R_2 - 3 R_1: \quad (3 - 3 \times 1) = 0, \quad (6 - 3 \times 2)=0, \quad (-3 + 3 \times 1)=0
$$

- Subtract 2 times row 1 from row 3:

$$
R_3 \leftarrow R_3 - 2 R_1: \quad (2 - 2 \times 1)=0, \quad (4 - 2 \times 2)=0, \quad (-2 + 2 \times 1)=0
$$

Resulting matrix:

$$
\begin{bmatrix}
1 & 2 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
$$

Step 3: Identify pivot variable:

- x_1 is a pivot variable.
- x_2 and x_3 are free variables.

Step 4: Express the solution:

From the first equation:

$$
x_1 + 2 x_2 - x_3 = 0 \Rightarrow x_1 = -2 x_2 + x_3
$$

Let:

$$
x_2 = s, \quad x_3 = t
$$

Then:

$$
x_1 = -2 s + t
$$

The general solution vector:

$$
\mathbf{x} = \begin{bmatrix}
x_1 \\ x_2 \\ x_3
\end{bmatrix}
= s \begin{bmatrix}
-2 \\ 1 \\ 0
\end{bmatrix}
+ t \begin{bmatrix}
1 \\ 0 \\ 1
\end{bmatrix}
$$

Step 5: Basis vectors:

$$
\boxed{
\left\{
\begin{bmatrix}
-2 \\ 1 \\ 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\ 0 \\ 1
\end{bmatrix}
\right\}
}
$$

These form a basis for the solution space. The dimension of the solution space (nullity) is 2, corresponding to the number of free variables.

Theoretical Foundations: Rank, Nullity, and the Rank-Nullity Theorem

A key theoretical result underpinning the process is the Rank-Nullity Theorem, which states:

$$
\text{rank}(A) + \text{nullity}(A) = n
$$

where:

- $\text{rank}(A)$ is the number of linearly independent rows (or columns) of A.
- $\text{nullity}(A)$ is the dimension of the solution space (null space).

This theorem provides a direct way to determine the dimension once the rank is known. For example, if the matrix A has rank r, then the solution space of the homogeneous system has dimension $n - r$.

Practical Strategies and Tools

In practice, solving these systems and finding bases can be streamlined using computational tools:

- Gaussian elimination remains the cornerstone technique.
- Matrix software such as MATLAB, NumPy (Python), or Wolfram Mathematica simplifies row operations and null space calculations.
- Online calculators provide quick solutions for small systems.

Additionally, understanding the structure of the matrix—such as recognizing block forms or sparsity—can optimize the process.

Conclusion: The Significance of Basis and Dimension

Finding a basis for the solution space of a homogeneous system and determining its dimension are central tasks

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