Find A Solution To The Initial Value Problem Y+sin(t)y=g(t), Y(0)=4, That Is Continuous On The Interval

Find A Solution To The Initial Value Problem Y+sin(t)y=g(t), Y(0)=4, That Is Continuous On The Interval is a classic problem in differential equations, specifically involving first-order linear differential equations. Such problems are fundamental in understanding how to model and solve dynamic systems across various fields like physics, engineering, and biology. In this article, we will explore the steps to find a solution to this initial value problem (IVP), analyze the conditions for continuity, and discuss the significance of the solution within its domain.

Understanding the Initial Value Problem (IVP)

The Differential Equation and Initial Condition

The given IVP is:

$$ Y + \sin(t) y = g(t), \quad Y(0) = 4 $$

Here, the notation can be clarified:
- $y = y(t)$ is the unknown function we are solving for.
- $g(t)$ is a given continuous function, defined on some interval around $t=0$.
- The initial condition specifies that at $t=0$, the solution $y(t)$ takes the value 4.

Note: The notation Y typically denotes the function $y(t)$; in this context, it is likely a typo, and the intended equation is:

$$ y + \sin(t) y = g(t) $$

or

$$ y' + \sin(t) y = g(t) $$

Given the structure, it seems more consistent that the differential equation is:

$$ y' + \sin(t) y = g(t) $$

and the initial condition $y(0) = 4$.

Assumption: To proceed logically, we assume the differential equation is:

$$ y' + \sin(t) y = g(t) $$

with $y(0) = 4$.

Transforming the Differential Equation into Standard Form

Recognizing a First-Order Linear ODE

The general form of a first-order linear ODE is:

$$ y' + p(t) y = q(t) $$

In our problem:

- $p(t) = \sin(t)$
- $q(t) = g(t)$

The solution approach involves finding an integrating factor, which simplifies the equation and allows us to integrate both sides.

Calculating the Integrating Factor

The integrating factor $\mu(t)$ is given by:

$$ \mu(t) = e^{\int p(t) dt} = e^{\int \sin(t) dt} $$

Calculating the integral:

$$ \int \sin(t) dt = -\cos(t) + C $$

Since the integrating factor involves an exponential, the constant C can be ignored in this context because it cancels out in the process.

Thus:

$$ \mu(t) = e^{-\cos(t)} $$

Solving the Differential Equation

Applying the Integrating Factor

Multiply the entire differential equation by $\mu(t)$:

$$e^{-\cos(t)} y' + e^{-\cos(t)} \sin(t) y = g(t) e^{-\cos(t)}$$

Observe that the left side is the derivative of $y(t) e^{-\cos(t)}$ with respect to t:

$$\frac{d}{dt} \left(y(t) e^{-\cos(t)} \right) = g(t) e^{-\cos(t)}$$

Integrating Both Sides

Integrate both sides from 0 to t:

$$y(t) e^{-\cos(t)} = y(0) e^{-\cos(0)} + \int_0^t g(s) e^{-\cos(s)} ds$$

Recall the initial condition:

$$y(0) = 4$$

and

$$\cos(0) = 1$$

Therefore:

$$y(t) e^{-\cos(t)} = 4 e^{-1} + \int_0^t g(s) e^{-\cos(s)} ds$$

Expressing the General Solution

To solve for $y(t)$:

$$y(t) = e^{\cos(t)} \left[4 e^{-1} + \int_0^t g(s) e^{-\cos(s)} ds \right]$$

This expression provides the explicit solution to the IVP, contingent on the function $g(t)$.

Ensuring Continuity of the Solution

Conditions for Continuity

For the solution $y(t)$ to be continuous on an interval containing $t=0$, the following conditions must be satisfied:

- The function $g(t)$ must be continuous on the interval.
- The exponential function $e^{\cos(t)}$ is continuous everywhere, so it poses no restrictions.
- The integral $\int_0^t g(s) e^{-\cos(s)} ds$ must be well-defined and continuous in t. This is guaranteed if $g(t)$ is continuous due to the Fundamental Theorem of Calculus.

Therefore, the solution $y(t)$ is continuous on any interval where $g(t)$ is continuous.

Summary of the Solution Process

- Identify the differential equation as linear and compute the integrating factor $\mu(t) = e^{-\cos(t)}$.

- Multiply through by the integrating factor to rewrite the equation as a total derivative.

- Integrate both sides and apply the initial condition to find the particular solution.

- Express the solution explicitly as:

  $$y(t) = e^{\cos(t)} \left[4 e^{-1} + \int_0^t g(s) e^{-\cos(s)} ds \right]$$

  - Verify the continuity of the solution based on the properties of $g(t)$.

Additional Considerations and Applications

Choosing the Interval

The solution's domain depends on the interval where $g(t)$ remains continuous. If $g(t)$ is continuous on $[a, b]$, then the solution $y(t)$ will be continuous on that interval.

Special Cases

- If $g(t)$ is zero, the solution reduces to:

$$y(t) = 4 e^{\cos(t) - 1}$$

- If $g(t)$ is a constant, say $g(t) = c$, then the integral becomes:

$$c \int_0^t e^{-\cos(s)} ds$$

and the solution simplifies accordingly.

Conclusion

Finding a solution to the initial value problem $y' + \sin(t) y = g(t)$, with $y(0)=4$, involves recognizing the type of differential equation, computing the integrating factor, integrating, and applying initial conditions. The explicit solution:

$$y(t) = e^{\cos(t)} \left[4 e^{-1} + \int_0^t g(s) e^{-\cos(s)} ds \right]$$

is continuous on any interval where $g(t)$ is continuous. This process exemplifies the power of integrating factors in solving linear differential equations and highlights the importance of understanding the conditions under which solutions are continuous and well-defined.

By mastering these techniques, students and practitioners can effectively solve a wide range of initial value problems, enabling accurate modeling and problem-solving across scientific disciplines.

Frequently Asked Questions

What is the initial value problem given for the differential equation?

The initial value problem is $Y + \sin(t) Y = g(t)$, with the initial condition $Y(0) = 4$.

How do we approach solving the differential equation $Y + \sin(t) Y = g(t)$?

We recognize it as a linear first-order differential equation and use an integrating factor to solve it.

What is the standard form of the differential equation in this problem?

The standard form is $Y' + P(t)Y = G(t)$, but here, it can be rewritten as $Y' + \sin(t) Y = g(t)$.

How do you find the integrating factor for this differential equation?

The integrating factor $\mu(t)$ is $e^{\int P(t)\,dt} = e^{\int \sin(t)\,dt} = e^{-\cos(t)}$.

What is the general solution to the differential equation?

The solution is $Y(t) = e^{-\cos(t)} [\int g(t) e^{\cos(t)}\,dt + C]$, where C is determined by the initial condition.

How do we determine the constant C using the initial condition Y(0)=4?

Plug t=0 into the general solution and Y(0)=4, then solve for C: $4 = e^{-\cos(0)} [\int g(t) e^{\cos(t)}\,dt$ from 0 to $0 + C]$, which simplifies to $4 = e^{-1} C$, so C = 4e.

Is the solution continuous on the interval, and why?

Yes, because the integrating factor and the integral are continuous functions, assuming g(t) is continuous, ensuring the solution Y(t) is continuous on the interval.

What conditions on g(t) are necessary for the solution to be valid and continuous?

g(t) must be continuous on the interval in question to ensure the integral and the solution Y(t) are continuous.

How can this method be applied to similar differential equations with different initial conditions?

The same process applies: find the integrating factor, solve the integral, and then use the initial condition to solve for the constant, ensuring the solution satisfies the new initial condition.

Additional Resources

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Introduction

The process of solving initial value problems (IVPs) in differential equations is fundamental to understanding dynamic systems across disciplines—be it physics, engineering, biology, or economics. Among the myriad of differential equations encountered, a class characterized by first-order linear equations holds particular importance due to their manageable structure and broad applicability.

This article embarks on a comprehensive investigation into solving the specific initial value problem:

$$ Y + \sin(t) y = g(t), \quad Y(0) = 4, $$

with an emphasis on constructing solutions that are continuous over an interval containing $t = 0$. The challenge here lies in identifying the explicit form of $y(t)$, ensuring continuity, and understanding the properties of the solution within the interval of interest.

Framing the Problem: Understanding the Differential Equation

The given IVP can be rewritten in standard form to facilitate solution strategies:

$$
Y + \sin(t) y = g(t),
$$

where Y is the derivative of y with respect to t, i.e.,

$$
Y = \frac{dy}{dt}.
$$

Thus, the differential equation becomes:

$$
\frac{dy}{dt} + \sin(t) y = g(t),
$$

with the initial condition:

$$
y(0) = 4.
$$

This is a linear first-order differential equation, a class well-understood in mathematical analysis. Its general form:

$$
\frac{dy}{dt} + p(t) y = q(t),

\]

has a well-known integrating factor approach for solution.

Theoretical Framework: Integrating Factor Method

The key to solving linear first-order equations is the integrating factor:

$$
\mu(t) = e^{\int p(t) dt}.
$$

In our case:

$$
p(t) = \sin(t),
$$

so the integrating factor becomes:

$$
\mu(t) = e^{\int \sin(t) dt} = e^{-\cos(t)}.
$$

This exponential form emerges because:

$$
\int \sin(t) dt = -\cos(t) + C,
$$

and the constant of integration can be ignored during the integrating factor calculation since it cancels out when multiplying through the differential equation.

Deriving the General Solution

Multiplying the entire differential equation by the integrating factor $e^{-\cos(t)}$:

$$
e^{-\cos(t)} \frac{dy}{dt} + e^{-\cos(t)} \sin(t) y = e^{-\cos(t)} g(t).
$$

Notice that the left side simplifies to the derivative of the product $y(t) e^{-\cos(t)}$:

\[

$$
\frac{d}{dt} \left(y(t) e^{-\cos(t)} \right) = e^{-\cos(t)} g(t).
$$

Integrating both sides with respect to t:

$$
y(t) e^{-\cos(t)} = \int e^{-\cos(t)} g(t) dt + C,
$$

where C is an arbitrary constant.

Finally, solving for $y(t)$:

$$
\boxed{
y(t) = e^{\cos(t)} \left(\int e^{-\cos(t)} g(t) dt + C \right).
}
$$

Applying the Initial Condition

To determine C, apply the initial condition $y(0) = 4$:

$$
4 = y(0) = e^{\cos(0)} \left(\int_{t_0=0} e^{-\cos(t)} g(t) dt \bigg|_{t=0} + C \right).
$$

Since:

$$
\cos(0) = 1,
$$

and noting the indefinite integral, we set:

$$
I(t) = \int_{0}^{t} e^{-\cos(s)} g(s) ds,
$$

then:

$$
4 = e^{1} (I(0) + C) = e \times (0 + C) \Rightarrow C = \frac{4}{e}.
$$

Hence, the explicit solution is:

$$

$$
\boxed{
y(t) = e^{\cos(t)} \left(\frac{4}{e} + \int_{0}^{t} e^{-\cos(s)} g(s) ds \right).
}
$$

Ensuring Continuity on the Interval

The key to establishing the solution's continuity hinges on the properties of the functions involved:

- $\cos(t)$ is continuous everywhere, so $e^{\cos(t)}$ is continuous everywhere.
- The integral $\int_{0}^{t} e^{-\cos(s)} g(s) ds$ depends on the properties of $g(t)$.

Conditions for continuity:

- If $g(t)$ is continuous on an interval I containing 0, then $e^{-\cos(s)} g(s)$ is continuous on I.
- The integral of a continuous function is continuous, so $I(t)$ is continuous on I.

Thus, if $g(t)$ is continuous on the interval, the solution $y(t)$ is continuous on that interval, including at $t=0$.

Special Cases and Practical Considerations

1. Known $g(t)$

If $g(t)$ is specified as a particular function, such as a polynomial, exponential, or sinusoid, the integral can be explicitly computed, leading to a closed-form solution. For example:

- If $g(t) = 0$, the solution simplifies to:

$$
y(t) = \frac{4}{e} e^{\cos(t)}.
$$

- If $g(t)$ is a constant g_0, then:

$$
I(t) = g_0 \int_{0}^{t} e^{-\cos(s)} ds,
$$

which might not have a closed-form solution but remains well-defined and continuous.

2. Numerical Approximation

When $g(t)$ lacks a closed-form integral, computational methods such as Simpson's rule or adaptive quadrature can approximate $I(t)$, preserving the solution's continuity.

Extended Analysis: Behavior and Stability

An in-depth investigation of the solution's behavior involves examining:

- Boundedness: Since $e^{\cos(t)}$ oscillates between e^{-1} and e^{1}, the solution's amplitude depends on the integral term and initial data.
- Continuity and Differentiability: Under the continuity of $g(t)$, the solution $y(t)$ is continuously differentiable, satisfying the original differential equation everywhere within the interval.

Conclusion and Summary

The initial value problem:

$$
\boxed{
Y + \sin(t) y = g(t), \quad y(0)=4,
}
$$

can be explicitly solved using the integrating factor method, yielding:

$$
\boxed{
y(t) = e^{\cos(t)} \left(\frac{4}{e} + \int_{0}^{t} e^{-\cos(s)} g(s) ds \right).
}
$$

Key insights:

- The solution is valid and continuous on any interval where $g(t)$ is continuous.
- The integral term's behavior directly influences the solution's qualitative properties.
- The method generalizes to any $g(t)$ satisfying the continuity condition, making the solution highly adaptable.

This investigation highlights the power and elegance of classical methods in solving linear differential equations, reinforcing their foundational role in mathematical analysis and applied sciences. Ensuring the continuity of \(g(t) \) guarantees a continuous solution, fulfilling the problem's core requirement.

References

- Boyce, W. E., & DiPrima, R. C. (2017). Elementary Differential Equations and Boundary Value Problems. Wiley.
- Zill, D. G. (2018). A First Course in Differential Equations. Cengage Learning.
- Tenenbaum, M., & Pollard, H. (1985). Ordinary Differential Equations. Dover Publications.

Note: The explicit form of the integral depends on the specific function \(g(t) \). When \(g(t) \) is known, further steps may involve direct computation or numerical approximation.

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