Find An Equation For The Surface Consisting Of All Points P For Which The Distance From P To The X-axis

Find An Equation For The Surface Consisting Of All Points P For Which The Distance From P To The X-axis is a classic problem in three-dimensional geometry that involves understanding the relationship between a point's location and its proximity to a specific axis. This problem invites us to explore the nature of surfaces generated by points satisfying certain distance criteria, which is fundamental in fields such as calculus, physics, and computer graphics. In this article, we will delve into how to derive the equation for such a surface, interpret its geometric significance, and understand related concepts.

Understanding the Problem: Points and Distances in 3D Space

Before jumping into the derivation, it is essential to clarify the problem's components:

- Point P: An arbitrary point in three-dimensional space, with coordinates (x, y, z).
- X-axis: The line in 3D space passing through the origin with direction vector $\mathbf{i} = (1, 0, 0)$.
- Distance from P to the X-axis: The shortest distance between the point P and the line representing the x-axis.

The goal is to find an explicit equation that describes all points $P = (x, y, z)$ such that their perpendicular distance to the x-axis equals a certain constant or, more generally, describes the entire surface formed by all such points.

Step 1: Expressing the Distance from a Point to the X-axis

Understanding how to compute the shortest distance from a point to the x-axis is the foundation for deriving the surface's equation.

Coordinates of Point P

Let $P = (x, y, z)$ be an arbitrary point in space.

The X-axis in Coordinates

The x-axis can be represented parametrically as all points $(t, 0, 0)$, where $t \in \mathbb{R}$.

Calculating the Shortest Distance

The shortest distance from P to the x-axis is the length of the perpendicular segment connecting P to some point on the x-axis. Since the x-axis is a line along the x-direction, this perpendicular segment will be orthogonal to the line.

Method:

- Find the point $Q = (t, 0, 0)$ on the x-axis such that the distance between P and Q is minimized.
- The distance d is then:

$$
d = \min_{t} \sqrt{(x - t)^2 + y^2 + z^2}
$$

- To find the value of t that minimizes this, differentiate with respect to t:

$$
f(t) = (x - t)^2 + y^2 + z^2
$$

$$
\frac{df}{dt} = 2(x - t)(-1) = -2(x - t)
$$

- Set derivative to zero for a minimum:

$$
-2(x - t) = 0 \quad \Rightarrow \quad t = x
$$

- Substitute back to find the closest point:

$$
Q = (x, 0, 0)
$$

- The shortest distance is then:

$$
d = \sqrt{(x - x)^2 + y^2 + z^2} = \sqrt{0 + y^2 + z^2} = \sqrt{y^2 + z^2}
$$

Result:

$$
\boxed{
\text{Distance from } P = (x, y, z) \text{ to the x-axis} = \sqrt{y^2 + z^2}
}
$$

This is a key insight: the distance depends only on the y and z coordinates, not on x.

Step 2: Formulating the Equation of the Surface

Having established that the distance from P to the x-axis is $\sqrt{y^2 + z^2}$, the problem reduces to describing the surface of all points P satisfying a specific distance condition.

Case 1: Fixed Distance r

Suppose we are interested in the surface consisting of all points $P = (x, y, z)$ such that the distance from P to the x-axis equals a constant $r \geq 0$:

$$
\sqrt{y^2 + z^2} = r
$$

Squaring both sides to eliminate the square root:

$$
y^2 + z^2 = r^2
$$

This equation describes a cylinder centered along the x-axis with radius r.

Geometric Interpretation

- For each fixed x, the cross-section of the surface is a circle in the yz-plane:

$$
y^2 + z^2 = r^2
$$

- As x varies over all real numbers, the entire surface is generated by translating this circle along the x-axis.

Therefore:

$$
\boxed{
\text{Equation of the surface:} \quad y^2 + z^2 = r^2
}
$$

This is a right circular cylinder aligned along the x-axis.

Case 2: Variable Distance d

If instead, we consider the set of points at a variable distance d, the general form becomes:

$$
y^2 + z^2 = d^2
$$

which represents a family of cylinders with different radii.

Step 3: General Equation for the Surface

The core of the problem is captured in the simple relationship:

$$
\boxed{
y^2 + z^2 = \text{constant}
}
$$

which defines a cylinder extending infinitely along the x-axis.

Thus, the general form of the surface is:

$$
\boxed{
\text{Surface } S: \quad y^2 + z^2 = R^2
}
$$

where R is a non-negative real number representing the radius of the cylinder.

Additional Considerations and Variations

While the previous sections describe the surface for a fixed radius R, other related surfaces can be considered:

1. Surfaces of Revolution

These are generated by revolving a circle around the x-axis, leading to cylindrical surfaces.

2. Surfaces with Variable Radius

If the radius varies with x, such as $R = R(x)$, the surface becomes a cylindrical surface with variable radius:

$$
y^2 + z^2 = R^2(x)
$$

for some function $R(x)$.

3. Distance from Other Axes

Similar reasoning applies if the distance is measured from other axes, such as the y-axis or z-axis, leading to different cylinders or surfaces.

Applications and Significance

Understanding the equation of surfaces based on distance criteria has numerous practical applications:

- Engineering: Designing cylindrical pipes, tunnels, or structural components.
- Physics: Modeling fields or regions with symmetry about an axis.
- Computer Graphics: Rendering objects with cylindrical symmetry.
- Mathematics: Studying surface properties, intersections, and geometric transformations.

Summary

To summarize, the key steps in deriving the equation for the surface consisting of all points $P = (x, y, z)$ at a fixed distance r from the x-axis are:

- Recognize that the shortest distance from P to the x-axis depends only on y and z: $d = \sqrt{y^2 + z^2}$.
- Set this distance equal to the constant r:

$$
\sqrt{y^2 + z^2} = r
$$

- Square both sides to obtain the standard form:

$$
y^2 + z^2 = r^2
$$

- The resulting equation describes a right circular cylinder extending infinitely along the x-axis.

This derivation exemplifies how geometric intuition and algebraic manipulation combine to describe complex surfaces succinctly.

Conclusion

Determining the equation of a surface based on a distance criterion from a fixed line or axis is a fundamental skill in geometry. In this case, the surface of all points at a fixed distance from the x-axis is a right circular cylinder described by the equation $y^2 + z^2 = R^2$. Understanding such surfaces enables mathematicians and engineers to analyze symmetrical shapes, design components, and explore spatial relationships effectively. Whether dealing with fixed or variable distances, the core approach involves translating geometric conditions into algebraic equations, revealing the elegant structure of three-dimensional space.

Frequently Asked Questions

What is the general approach to find the equation of a surface consisting of all points P that are a fixed distance from the x-axis?

The approach involves setting the perpendicular distance from a generic point P(x, y, z) to the x-axis equal to a constant and then expressing this as an equation. Since the distance from P to the x-axis depends on y and z, the equation typically takes the form $y^2 + z^2 = r^2$, where r is the fixed distance.

How do you derive the equation of the surface for points at a fixed distance r from the x-axis?

The distance from a point P(x, y, z) to the x-axis is given by $\sqrt{(y^2 + z^2)}$. Setting this equal to a constant r yields the equation $y^2 + z^2 = r^2$, which describes a cylindrical surface centered along the x-axis.

What type of surface is formed by all points at a fixed distance from the x-axis?

The surface is a right circular cylinder with its axis along the x-axis, with radius equal to the fixed distance r. It extends infinitely along the x-axis.

How does the equation change if the distance from P to the x-axis is not constant but varies along the x-axis?

If the distance varies along the x-axis, say as a function r(x), then the equation becomes $y^2 + z^2 = [r(x)]^2$, creating a surface of varying radius along the x-axis, which could form a cone or a more complex shape depending on r(x).

Can the surface be described in terms of cylindrical coordinates?

Yes. Using cylindrical coordinates with respect to the y-z plane, where $y = \rho \cos \theta$ and $z = \rho \sin \theta$, the surface is described by $\rho = r$, a constant, for all x, θ, which confirms it's a cylinder of radius r extending infinitely along the x-axis.

What is the significance of the surface's symmetry in this context?

The surface exhibits rotational symmetry about the x-axis, because the equation $y^2 + z^2 = r^2$ is independent of the angle θ, indicating the surface is a perfect cylinder extending infinitely along the x-axis.

How would you modify the equation if the fixed distance is expressed as a variable function of x, say d(x)?

You would replace the constant r with the function d(x), resulting in the equation $y^2 + z^2 = [d(x)]^2$, describing a surface where the radius of the cylinder varies along the x-axis.

Additional Resources

Finding the Equation of the Surface Consisting of All Points P for Which the Distance From P to the X-Axis

Understanding the geometric nature of surfaces in three-dimensional space is fundamental to advanced mathematics, particularly in calculus and analytic geometry. One common problem involves determining the equation of a surface defined by a specific distance condition relative to a given line or axis. In this case, our goal is to find the equation of the surface composed of all points \(P \) such that the distance from \(P \) to the x-axis is a specific value or variable. This exploration leads us into the concepts of distance formulas, surfaces of revolution, and the geometric interpretation of these surfaces.

Understanding the Geometric Setup

Before deriving the equation, it's essential to grasp the geometric scenario:

- Points in 3D space: Each point \(P \) can be represented as \((x, y, z) \).
- The x-axis: This is the line in 3D space where \(y = 0 \) and \(z = 0 \) for all \(x \). It extends infinitely in both directions along the x-axis.
- Distance from a point to the x-axis: The shortest path from \(P \) to the x-axis is the perpendicular distance from \(P \) to the line \(y = 0, z = 0 \).

Mathematical Foundations

To find the surface, several fundamental concepts are involved:

1. Distance from a Point to a Line in 3D

The shortest distance from a point \(P = (x, y, z) \) to a line \(L \) (here, the x-axis) is given by the formula:

\[
\text{Distance} = \frac{\| \vec{P} - \text{Projection of } \vec{P} \text{ onto } L \|}{\text{Line direction vector}}
\]

But more straightforwardly, since the line is the x-axis, the distance from \(P \) to the x-axis depends only on \(y \) and \(z \):

\[
\text{Distance} = \sqrt{(y - 0)^2 + (z - 0)^2} = \sqrt{y^2 + z^2}

\]

This simplifies the problem because the x-coordinate does not influence the distance to the x-axis.

2. Surface Definition Based on Distance

Suppose the distance from any point \(P = (x, y, z) \) to the x-axis is some constant \(r \). Then, the set of all such points forms a surface:

\[
\sqrt{y^2 + z^2} = r
\]

When \(r \) varies over all positive real numbers, the union of all these surfaces forms a three-dimensional set.

Deriving the Equation of the Surface

1. Fixed Distance Surface: Cylinders

- For a fixed value of \(r \), the set of points \(P \) satisfying

\[
\sqrt{y^2 + z^2} = r
\]

corresponds to a circular cylinder aligned along the x-axis.

- Equation of the cylinder:

\[
y^2 + z^2 = r^2
\]

- Interpretation:

This is an infinite circular cylinder centered along the x-axis with radius \(r \). The x-coordinate is unrestricted because the distance to the x-axis depends only on \(y \) and \(z \).

2. Surface of All Points with Variable Distance \(r \)

- If the problem asks for the entire surface of points where the distance from \(P \) to the x-axis is any value (not fixed), then the surface is the union of all such cylinders for \(r \geq 0 \).

- This union results in a cylindrical surface extending infinitely along the x-axis, with circular cross-sections of varying radius \(r \). Essentially, it's the collection of all points in space for which:

$$
\sqrt{y^2 + z^2} \geq 0
$$

- Note: For a specific r, the equation is:

$$
\boxed{ y^2 + z^2 = r^2 }
$$

- For the entire surface (all $r \geq 0$), it is described by:

$$
\text{The set of all } (x, y, z) \text{ such that } y^2 + z^2 \geq 0
$$

which is the entire space minus the x-axis itself (since for $r = 0$), the surface reduces to the x-axis itself).

Special Cases and Variations

1. Fixed Distance: The Cylinder Equation

When the distance from P to the x-axis is a constant r, the surface is a cylinder:

$$
\boxed{ y^2 + z^2 = r^2 }
$$

- Characteristics:

- Infinite along the x-axis.
- Circular cross-section perpendicular to the x-axis.
- Radius r.

- Graphical interpretation: Imagine a pipe extending infinitely in both directions along the x-axis, with a fixed radius r.

2. Variable Distance (Entire Surface)

If the distance varies over all non-negative values, the surface is the union of all cylinders with varying radii, which geometrically is equivalent to the entire space except the x-axis itself.

3. Surface of Revolution Perspective

The surface described can be viewed as a surface of revolution generated by revolving a circle $y^2 + z^2 = r^2$ around the x-axis. This is a classical surface in geometry, known as a cylinder.

Visualization and Geometric Intuition

Understanding the surface geometrically helps in visualizing the problem:

- Cylinder centered on the x-axis: For a fixed radius r, the surface is an infinite cylinder with circular cross-sections perpendicular to the x-axis.

- For all $r \ge 0$: The union of all such cylinders fills the entire space except for the x-axis itself.

- Special case $r=0$: The set reduces to the x-axis, which is a line in space.

- Implication: The entire "family" of these cylinders describes a cylindrical "shell" that extends infinitely along the x-axis.

Mathematical Summary and Final Equation

The core result derived from the geometric and algebraic analysis is:

$$
\boxed{
\text{Surface consisting of all points } P = (x, y, z) \text{ such that } y^2 + z^2 = r^2, \quad r \geq 0
}
$$

In particular, if the distance from P to the x-axis is a variable (say, r), then the general form of the surface is:

$$
\boxed{
\text{All points } (x, y, z) \text{ satisfying } y^2 + z^2 \leq R^2
}
$$

where R is some maximum radius, or simply considering the union over all r, the entire space minus the x-axis.

Applications and Further Insights

Understanding this surface has several applications:

- Engineering and design: Cylindrical structures, pipelines, and tunnels often involve surfaces described by such equations.

- Physics: Fields such as electromagnetism and fluid dynamics involve surfaces equidistant from a line or axis.

- Mathematics: Analyzing the properties of cylinders and surfaces of revolution deepens understanding of surface topology, symmetry, and calculus.

- Computational geometry: Algorithms for rendering 3D models often rely on implicit surface equations like these.

Extensions and Related Problems

The concepts discussed can be extended or modified for different axes and distance functions:

- Distance from other lines or points: For example, the set of points at a fixed distance from the y-axis or a point (a, b, c).

- Different distance functions: Using Manhattan distance or other metrics to define more complex surfaces.

- Surfaces based on other geometric constraints: Such as paraboloids, hyperboloids, or ellipsoids, which involve quadratic equations.

Conclusion

In summary, determining the equation of the surface composed of all points $P = (x, y, z)$ with a fixed or variable distance from the x-axis boils down to understanding the geometry of cylinders in three dimensions. The key insight is that the perpendicular distance from a point to the x-axis depends solely on the y and z coordinates, leading to the fundamental equation:

$$
\boxed{
y^2 + z^2 = r^2
}
$$

for a fixed radius \(r \), which describes a circular cylinder aligned along the x-axis. When considering all possible distances \(r \), the resulting surface encompasses an entire family of cylinders or, in the case of variable \(r \), the entire space (excluding the x-axis).

This exploration showcases the elegant interplay between algebra, geometry, and

Find An Equation For The Surface Consisting Of All Points P For Which The Distance From P To The X Axis

Find An Equation For The Surface Consisting Of All Points P For Which The Distance From P To The X Axis

Related Articles

- Exercice 3-(4 Pts): Enchanements D'oprations, Simplification De Fractions Et Nombres Relatifs. Recopier
- Formaggio Vecchio Announced Its Regular Quarterly Cash Dividend Of $0.20 Per Share. Currently There Are
- If A Center Employs More Than ____ People, Certain Written Policies Must Be In Place And Discussed With

Back to Home