Find The General Solution Of The Following Differential Equations:d^4y/dx^4 + 6 D^3y/dx^3 + 9 D^2y/dx^2

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Introduction

Differential equations play a vital role in modeling various physical phenomena, engineering systems, and scientific processes. Among these, linear differential equations with constant coefficients form a fundamental class that can be systematically solved using characteristic equations. The given differential equation,

$$
\frac{d^4 y}{d x^4} + 6 \frac{d^3 y}{d x^3} + 9 \frac{d^2 y}{d x^2} = 0,
$$

represents a linear homogeneous differential equation of the fourth order with constant coefficients. The goal of this article is to derive its general solution comprehensively, exploring the underlying theory, solving step-by-step, and discussing the nature of the solutions obtained.

Understanding the Differential Equation

The differential equation involves derivatives up to the fourth order, with coefficients that are constants. It can be rewritten for clarity as:

$$
D^4 y + 6 D^3 y + 9 D^2 y = 0,
$$

where $D^n y$ indicates the n^th derivative of y with respect to x.

Key observations:
- The equation is linear and homogeneous.
- All terms involve derivatives of y, with coefficients being constants.
- The highest derivative order is four, which indicates the general solution will involve four arbitrary constants.

Methodology for Solving the Differential Equation

Characteristic Equation Approach

The most straightforward method to solve such linear equations with constant coefficients is to assume solutions of the form:

$$
y = e^{rx},
$$

where r is a constant to be determined. Substituting into the differential equation converts it into an algebraic polynomial equation known as the characteristic (or auxiliary) equation.

Steps Involved

- Replace derivatives with powers of r: $D^n y \rightarrow r^n e^{rx}$.
- Form the characteristic polynomial.
- Find the roots of the polynomial.
- Construct the general solution based on the roots.

Forming the Characteristic Equation

Assuming $y = e^{rx}$, then:

$$
D^4 y = r^4 e^{rx}, \quad D^3 y = r^3 e^{rx}, \quad D^2 y = r^2 e^{rx}.
$$

Substituting into the differential equation:

$$
r^4 e^{rx} + 6 r^3 e^{rx} + 9 r^2 e^{rx} = 0.
$$

Dividing through by e^{rx} (which is never zero):

$$
r^4 + 6 r^3 + 9 r^2 = 0.
$$

This polynomial equation:

$$
r^4 + 6 r^3 + 9 r^2 = 0,
$$

is the characteristic equation corresponding to the differential equation.

Solving the Characteristic Equation

Factorization

First, factor out the common term r^2:

$$
r^2 (r^2 + 6 r + 9) = 0.
$$

This yields:

$$
r^2 = 0,
$$

and

$$
r^2 + 6 r + 9 = 0.
$$

Roots of the Characteristic Equation

1. Root from $r^2 = 0$:

$$
r = 0, \quad \text{with multiplicity 2}.
$$

2. Root from quadratic $r^2 + 6 r + 9 = 0$:

Solve using quadratic formula:

$$
r = \frac{-6 \pm \sqrt{36 - 36}}{2} = \frac{-6 \pm 0}{2} = -3.
$$

Since discriminant is zero, this root has multiplicity 2:

$$
r = -3, \quad \text{with multiplicity 2}.
$$

Constructing the General Solution

Root Types and Corresponding Solutions

- Real distinct roots: r.
- Repeated roots: include polynomial factors multiplied by x to account for multiplicity.

Given roots:

- $r = 0$ (double root),
- $r = -3$ (double root).

The general solution for a linear differential equation with these roots involves:

$$
y(x) = \sum \text{(solutions corresponding to each root)}.
$$

Specifically:

- For a simple root r, the solution is e^{rx}.
- For a repeated root with multiplicity m, solutions include e^{rx}, $x e^{rx}$, ..., $x^{m-1} e^{rx}$.

Applying this:

$$
\boxed{
y(x) = C_1 e^{0 \cdot x} + C_2 x e^{0 \cdot x} + C_3 e^{-3x} + C_4 x e^{-3x},
}
$$

which simplifies to:

$$
\boxed{
y(x) = C_1 + C_2 x + C_3 e^{-3x} + C_4 x e^{-3x},
}
$$

where C_1, C_2, C_3, C_4 are arbitrary constants determined by initial conditions.

Summary of the General Solution

The general solution to the differential equation

$$
\frac{d^4 y}{d x^4} + 6 \frac{d^3 y}{d x^3} + 9 \frac{d^2 y}{d x^2} = 0,
$$

is:

\[
\boxed{

y(x) = C_1 + C_2 x + C_3 e^{-3x} + C_4 x e^{-3x}.
}
\]

This solution encompasses all possible solutions of the differential equation, parameterized by four arbitrary constants.

Additional Remarks and Insights

Nature of the Roots and Solution Behavior

- The roots 0 and -3 are real, with multiplicities greater than one.
- The exponential terms e^{-3x} and $x e^{-3x}$ describe solutions that decay exponentially as $x \to \infty$.
- The polynomial solutions $C_1 + C_2 x$ correspond to the solutions associated with the zero roots, representing linear behavior.

Implications in Physical Systems

- The combination of polynomial and exponential solutions reflects systems with modes that either grow linearly or decay exponentially, common in mechanical vibrations, electrical circuits, or thermal processes.

Extensions and Applications

- Initial conditions can be used to determine the specific values of C_1, C_2, C_3, and C_4.
- Similar methodology applies to higher-order linear differential equations with constant coefficients, emphasizing the importance of characteristic equations.

Conclusion

The process of solving the differential equation involved transforming it into its characteristic polynomial, factoring the polynomial to find roots, and constructing the general solution using the roots' multiplicities. Understanding the roots' nature informs us of the solution's structure, which combines polynomial and exponential functions. This approach underscores the power and elegance of the characteristic equation method in solving linear differential equations with constant coefficients, providing comprehensive insight into the behavior of such systems.

Frequently Asked Questions

What is the general solution to the differential equation d^4y/dx^4 + 6 d^3y/dx^3 + 9 d^2y/dx^2 = 0?

The general solution involves solving its characteristic equation, which is $r^4 + 6r^3 + 9r^2 = 0$. Factoring out r^2 gives $r^2(r^2 + 6r + 9) = 0$. This yields roots $r = 0$ (double root) and $r = -3$ (double root), leading to the general solution $y(x) = C_1 + C_2 x + C_3 e^{-3x} + C_4 x e^{-3x}$, where C_1, C_2, C_3, C_4 are arbitrary constants.

How do you find the roots of the characteristic equation for the differential equation d^4y/dx^4 + 6 d^3y/dx^3 + 9 d^2y/dx^2 = 0?

First, write the characteristic equation: $r^4 + 6 r^3 + 9 r^2 = 0$. Factor out r^2: $r^2(r^2 + 6 r + 9) = 0$. Then, solve $r^2 = 0$ giving roots $r = 0$ (double root), and solve $r^2 + 6 r + 9 = 0$, which factors as $(r + 3)^2 = 0$, giving roots $r = -3$ (double root).

What is the significance of multiple roots in the characteristic equation when solving differential equations?

Multiple roots indicate repeated roots, which lead to solutions involving polynomial factors multiplied by exponential functions. For a root r with multiplicity m, the solutions include terms like $x^{k} e^{r x}$ for $k = 0$ to $m-1$, ensuring the general solution accounts for multiplicity.

Can you explain the method to derive the general solution from the characteristic roots for this differential equation?

Yes. After finding the roots of the characteristic equation, each root corresponds to a term in the solution. Simple roots r produce $e^{r x}$ terms. Multiple roots produce terms multiplied by powers of x. For complex roots, solutions involve exponential and sinusoidal functions. Combining all these, the general solution is a linear combination of these fundamental solutions.

What types of solutions appear if the characteristic roots are repeated in the differential equation?

Repeated roots result in solutions that include polynomial factors times exponential functions. For example, if r is a root of multiplicity m, the solutions include terms like $e^{r x}$, $x e^{r x}$, $x^2 e^{r x}$, up to $x^{m-1} e^{r x}$.

Is the differential equation d^4y/dx^4 + 6 d^3y/dx^3 + 9 d^2y/dx^2 = 0 homogeneous?

Yes, it is a homogeneous linear differential equation with constant coefficients, since all terms involve derivatives of y and there is no additional nonhomogeneous term.

How do the roots of the characteristic equation influence the form of the general solution?

The roots determine the types of functions in the solution. Real distinct roots lead to exponential solutions, repeated roots lead to polynomial multiplied exponentials, and complex roots lead to sinusoidal functions combined with exponentials. In this case, the roots $r=0$ (double) and $r=-3$ (double) produce polynomial and exponential solutions accordingly.

Additional Resources

Find The General Solution Of The Following Differential Equations: $d^4y/dx^4 + 6 D^3y/dx^3 + 9 D^2y/dx^2$

Introduction

Differential equations are fundamental in describing various phenomena across physics, engineering, economics, and many other scientific disciplines. Among these, linear differential equations with constant coefficients hold a special place due to their well-understood solution methods and wide applicability.

In this discussion, we focus on a specific linear differential equation:

$$
\frac{d^4 y}{dx^4} + 6 \frac{d^3 y}{dx^3} + 9 \frac{d^2 y}{dx^2} = 0
$$

The goal is to find its general solution, which encompasses all possible solutions satisfying the differential equation. This involves several steps, including identifying the characteristic equation, solving it, and constructing the general solution based on its roots.

Understanding the Differential Equation

The given differential equation is a linear, homogeneous differential equation with constant coefficients of the fourth order. It can be expressed as:

$$
D^4 y + 6 D^3 y + 9 D^2 y = 0
$$

where $D = \frac{d}{dx}$.

This type of differential equation is fundamental because the superposition principle applies, allowing the general solution to be expressed as a linear combination of basis solutions derived from the roots of the characteristic equation.

Step 1: Formulating the Characteristic Equation

The first step in solving such a differential equation is to convert it into an algebraic characteristic (or auxiliary) equation. To do this:

- Replace each derivative $\frac{d^n y}{dx^n}$ with r^n.
- The differential equation becomes a polynomial in r:

$$
r^4 + 6 r^3 + 9 r^2 = 0
$$

This polynomial is called the characteristic equation.

Step 2: Solving the Characteristic Equation

The characteristic equation is:

$$
r^4 + 6 r^3 + 9 r^2 = 0
$$

Factoring out the common term r^2:

$$
r^2 (r^2 + 6 r + 9) = 0
$$

This gives two factors to analyze separately:

1. $r^2 = 0$
2. $r^2 + 6 r + 9 = 0$

Solution for $r^2 = 0$:

$$
r = 0 \quad \text{(double root)}
$$

Solution for $r^2 + 6 r + 9 = 0$:

This is a quadratic equation. Solving using the quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

where $a = 1$, $b = 6$, $c = 9$.

Calculate discriminant:

$$
\Delta = 6^2 - 4 \times 1 \times 9 = 36 - 36 = 0
$$

Since the discriminant is zero, the roots are repeated real roots:

$$
r = \frac{-6 \pm 0}{2} = -3

\]

which is a double root.

Step 3: Roots of the Characteristic Equation

Summarizing:

- $r = 0$ (multiplicity 2)
- $r = -3$ (multiplicity 2)

This set of roots completely characterizes the solution space of the differential equation.

Step 4: Constructing the General Solution

The general solution to a linear differential equation with constant coefficients depends on the roots of the characteristic equation:

- For each distinct real root r, the solution includes a term e^{rx}.
- For repeated roots with multiplicity m, the solutions include terms $x^k e^{rx}$ for $k = 0, 1, ..., m-1$.

Given our roots:

Root	Multiplicity	Corresponding Solution Term
0	2	$e^{0 \cdot x} = 1$ and $x e^{0 \cdot x} = x$
-3	2	e^{-3x} and $x e^{-3x}$

Thus, the general solution is:

$$
\boxed{
y(x) = C_1 + C_2 x + C_3 e^{-3x} + C_4 x e^{-3x}
}
$$

where C_1, C_2, C_3, C_4 are arbitrary constants determined by initial or boundary conditions.

Deep Dive: Understanding the Components of the Solution

1. The C_1 and $C_2 x$ Terms

These arise from the roots at $r=0$. The constant term C_1 corresponds to the solution component associated with the root $r=0$ with multiplicity 1 (or 2, accounting for the repeated root). The x term accounts for the multiplicity of the root.

2. The $C_3 e^{-3x}$ and $C_4 x e^{-3x}$ Terms

These are associated with the root $r = -3$, which is a repeated root. The exponential decay term e^{-3x} reflects the negative root, and the $x e^{-3x}$ term accounts for the multiplicity.

Special Cases and Variations

While the solution presented is for the homogeneous differential equation, real-world problems sometimes involve nonhomogeneous terms. In such cases, the method of undetermined coefficients or variation of parameters could be employed to find particular solutions.

Additionally, if the roots had complex parts, the solutions would involve sinusoidal functions, as per Euler's formula, but in this case, all roots are real.

Additional Considerations

1. Initial Conditions

The general solution contains four arbitrary constants (C_1, C_2, C_3, C_4), which can be fixed if initial conditions are provided:

- $y(x_0) = y_0$
- $y'(x_0) = y_1$
- $y''(x_0) = y_2$
- $y'''(x_0) = y_3$

Applying these conditions involves differentiating the general solution accordingly and solving the resulting system for the constants.

2. Stability and Behavior of the Solution

The exponential terms e^{-3x} decay as $x \to \infty$, indicating damping or stabilization in physical systems modeled by this differential equation. The polynomial terms x and constant terms reflect potential steady-state or unchanging behaviors.

Practical Applications

The differential equation in question could model various physical systems, such as:

- Mechanical vibrations where higher derivatives represent acceleration, jerk, and higher-order effects.
- Electrical circuits involving inductance and capacitance with complex feedback.
- Structural dynamics with complex damping and stiffness characteristics.

Understanding the general solution helps engineers and scientists predict system behavior under different initial conditions, design control strategies, and analyze stability.

Summary

To summarize the process of solving the differential equation:

1. Identify the differential equation: Recognize it as a linear, homogeneous, constant-coefficient differential equation.
2. Formulate the characteristic equation: Replace derivatives with powers of r.
3. Solve the characteristic polynomial: Factor to find roots and their multiplicities.
4. Construct the general solution: Use exponential and polynomial terms based on roots.
5. Express the final form: Incorporate arbitrary constants for generality.

The final answer:

$$
\boxed{
\textbf{General solution:} \quad y(x) = C_1 + C_2 x + C_3 e^{-3x} + C_4 x e^{-3x}
}
$$

This comprehensive approach demonstrates the power of characteristic equations in solving linear differential equations and emphasizes the importance of understanding root multiplicities and their impact on the solution's form.

Closing Remarks

Understanding and solving differential equations like the one discussed is foundational for analyzing complex systems. Mastery of characteristic equations, roots, and solution construction paves the way for tackling more intricate equations involving variable coefficients, nonhomogeneous terms, or systems of differential equations. Whether in physics, engineering, or applied mathematics, these techniques are essential tools for modeling and understanding the natural and engineered world.

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