Find The General Solution Of The Given Higher-order Differential Equation. Y+2y16y32y = 0 Y(x) = _______

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Introduction

Differential equations form a fundamental part of mathematical modeling, describing phenomena across physics, engineering, biology, and many other sciences. Among these, higher-order differential equations—those involving derivatives of order greater than one—are particularly significant because they often model complex systems with multiple degrees of freedom. In this article, we focus on solving a specific higher-order differential equation:

Y + 2y'' + 16y''' + 32y□□□ = 0

Our goal is to find the general solution of this differential equation, which includes all possible solutions expressed in terms of arbitrary constants.

Understanding the Differential Equation

The Given Equation

The differential equation under consideration is:

$Y + 2y'' + 16y''' + 32y^{(4)} = 0$

where:

- $y = y(x)$ is the unknown function of x
- y'', y''', $y^{(4)}$ are the second, third, and fourth derivatives of y with respect to x

Recognizing the Type of Differential Equation

This is a linear homogeneous differential equation with constant coefficients and derivatives up to the fourth order. Such equations are generally solved using the characteristic equation method.

Step-by-Step Solution Process

Step 1: Write the Differential Equation in Standard Form

Express the equation explicitly:

$$32 y^{(4)} + 16 y''' + 2 y'' + y = 0$$

or equivalently,

$$32 y^{(4)} + 16 y''' + 2 y'' + y = 0$$

Step 2: Assume a Solution of the Form $y = e^{rx}$

This is a standard approach for linear differential equations with constant coefficients. Assume:

$$ y = e^{rx} $$

where r is a constant to be determined.

Step 3: Derive the Characteristic Equation

Substitute $y = e^{rx}$ into the differential equation:

$$ 32 r^4 e^{rx} + 16 r^3 e^{rx} + 2 r^2 e^{rx} + e^{rx} = 0 $$

Divide through by e^{rx} (which is never zero):

$$ 32 r^4 + 16 r^3 + 2 r^2 + 1 = 0 $$

This is the characteristic equation:

$$ 32 r^4 + 16 r^3 + 2 r^2 + 1 = 0 $$

Solving the Characteristic Equation

Step 4: Analyze the Polynomial

The characteristic polynomial:

$$ 32 r^4 + 16 r^3 + 2 r^2 + 1 = 0 $$

is a quartic equation with real coefficients. To solve it, look for possible rational roots using Rational Root Theorem or factorization techniques.

Step 5: Rational Root Test

Possible rational roots are factors of the constant term (±1) divided by factors of the leading coefficient (±1, ±2, ±4, ±8, ±16, ±32). Testing $r = \pm 1/2$, $\pm 1/4$, etc., is advisable.

- Test $r = -1/2$:

$$32 \left(-\frac{1}{2}\right)^4 + 16 \left(-\frac{1}{2}\right)^3 + 2 \left(-\frac{1}{2}\right)^2 + 1$$

Calculate step-by-step:

$$32 \times \frac{1}{16} + 16 \times \left(-\frac{1}{8}\right) + 2 \times \frac{1}{4} + 1$$

$$2 - 2 + 0.5 + 1 = 1.5 \neq 0$$

- Test $r = -1/4$:

$$32 \times \frac{1}{256} + 16 \times \left(-\frac{1}{64}\right) + 2 \times \frac{1}{16} + 1$$

$$0.125 - 0.25 + 0.125 + 1 = 1 \neq 0$$

Similarly, testing other rational roots yields no zero. Therefore, the roots are irrational or complex, and factoring directly is cumbersome.

Step 6: Use Polynomial Factoring Techniques or Numerical Methods

Since rational roots are not apparent, employ substitution or quadratic factorization methods.

Factoring the Characteristic Polynomial

Step 7: Rewrite as a Quadratic in r^2

Express as:

$$32 r^4 + 16 r^3 + 2 r^2 + 1 = 0$$

Group as:

$$(32 r^4 + 16 r^3) + (2 r^2 + 1)$$

Factor:

$$16 r^3 (2 r + 1) + 1 (2 r^2 + 1)$$

This doesn't immediately help, so alternatively, attempt substitution:

Let $s = r^2$, then:

$$32 s^2 + 16 r^3 + 2 s + 1 = 0$$

But since $r^3 = r \times r^2 = r \times s$, this introduces mixed terms; thus, this approach complicates matters.

Step 8: Numerical Approximation or Use of Computational Tools

Given the complexity, employing computational tools (like a graphing calculator, WolframAlpha, or software like MATLAB or Mathematica) can find approximate roots of the polynomial.

Suppose we find roots:

- Two complex conjugate roots: $r = \alpha \pm i \beta$

- Two real roots: $r = r_1$ and $r = r_2$

The exact roots are not necessary for the conceptual understanding; what matters is that the roots will determine the general solution.

Constructing the General Solution

Step 9: Write the General Solution Based on Roots

- For each distinct real root r: the solution includes e^{rx}.
- For complex conjugate roots $\alpha \pm i \beta$: the solution includes $e^{\alpha x}$ multiplied by sine and cosine functions:

$$y_{complex} = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

where C_1, C_2 are arbitrary constants.

Step 10: Final Form of the General Solution

Assuming roots r_1, r_2 (real) and conjugate pair $\alpha \pm i \beta$, the general solution is:

$$
\boxed{
y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + e^{\alpha x} \left(C_3 \cos \beta x + C_4 \sin \beta x \right)
}
$$

where C_1, C_2, C_3, C_4 are arbitrary constants determined by initial conditions.

Summary of the Solution Methodology

1. Identify the type of differential equation: linear homogeneous with constant coefficients.
2. Write the characteristic equation by substituting $y = e^{rx}$.
3. Solve the characteristic polynomial for roots:
   - Rational roots via Rational Root Theorem (if possible).
   - Use numerical or algebraic software for complex roots.
4. Construct the general solution based on the roots:
   - Real roots: exponential functions.
   - Complex roots: exponential times sine/cosine functions.
5. Combine all solutions to form the most general solution with arbitrary constants.

Additional Notes on Higher-Order Differential Equations

Homogeneous vs Non-Homogeneous Equations

- The discussed method applies to homogeneous equations. For non-homogeneous equations, particular solutions are added to the complementary solution.

Initial Conditions

- To determine the arbitrary constants C_1, C_2, C_3, C_4, initial conditions such as $y(0), y'(0), y''(0), y'''(0)$ are needed.

Applications

- Such differential equations appear in mechanical vibrations, electrical circuits, and other dynamic systems.

Conclusion

Finding the general solution of higher-order differential equations involves understanding the characteristic equation, solving for roots, and constructing the solution accordingly. For the specific differential equation:

$$ y + 2 y'' + 16 y''' + 32 y^{(4)} = 0 $$

the solution process is systematic but may involve complex roots requiring computational tools. Nonetheless, the fundamental approach remains consistent: assume exponential solutions, derive the characteristic polynomial, solve for roots, and then form the general solution based on these roots.

References

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- Zill, D. G. (2010). Differential Equations with Boundary-Value Problems. Brooks Cole.

Frequently Asked Questions

What is the general approach to solving higher-order differential equations like $Y + 2Y' + 16Y'' + 32Y''' = 0$?

The standard approach is to find the characteristic equation associated with the differential equation, solve for its roots, and then write the general solution based on the roots (real, repeated, or complex).

How do you form the characteristic equation for the differential equation Y + 2Y' + 16Y'' + 32Y''' = 0?

Replace Y with r, Y' with r, Y'' with r^2, and Y''' with r^3, leading to the characteristic equation: r^3 + 16r^2 + 2r + 1 = 0.

What are the roots of the characteristic equation r^3 + 16r^2 + 2r + 1 = 0?

By solving the cubic, we find the roots are r = -1 (repeated), and r = -16, or complex roots depending on the factorization. The exact roots depend on factorization or numerical methods.

How do repeated roots affect the form of the general solution?

If a root r is repeated n times, the terms in the general solution include polynomial factors: for a double root r, terms are e^{r x} and x e^{r x}.

What is the general solution form if the roots are r = -1 (double root) and r = -16?

The general solution would be: Y(x) = C1 e^{-16x} + C2 e^{-x} + C3 x e^{-x}, where C1, C2, and C3 are arbitrary constants.

How can you verify the correctness of the obtained general solution?

By differentiating the general solution and substituting back into the original differential equation to ensure it satisfies the equation for all x.

Are there special cases where the roots are complex, and how does that change the solution?

Yes, if roots are complex conjugates, the solution involves exponential and trigonometric functions: e^{□x}(A cos □x + B sin □x).

What is the final answer to the given differential equation Y + 2Y' + 16Y'' + 32Y''' = 0?

The general solution is Y(x) = C1 e^{-16x} + C2 e^{-x} + C3 x e^{-x}, where C1, C2, and C3 are arbitrary constants.

Additional Resources

Find The General Solution Of The Given Higher-order Differential Equation. Y + 2y' + 16y'' + 32y''' = 0, Y(x) = ______

Understanding how to find the general solution of higher-order differential equations is fundamental in calculus and differential equations, especially when dealing with real-world phenomena such as mechanical vibrations, electrical circuits, and dynamic systems. The problem at hand involves a linear differential equation of the third order with constant coefficients. This type of equation is a classic example in differential equations courses and requires a systematic approach involving characteristic equations. In this comprehensive review, we will dissect the problem step by step, explore methods to solve it, and analyze the features and implications of the solutions obtained.

Understanding the Differential Equation

The differential equation provided is:

$$ Y + 2 y' + 16 y'' + 32 y''' = 0 $$

where Y and y are functions of x. Assuming the notation, it appears that Y represents the function $y(x)$, making the equation:

$$ y + 2 y' + 16 y'' + 32 y''' = 0 $$

This is a third-order linear differential equation with constant coefficients. The order of the differential equation is determined by the highest derivative present—in this case, the third derivative y'''.

Key features:

- Linearity: The equation is linear because each term involves only y or its derivatives to the first power.
- Constant coefficients: The coefficients (1, 2, 16, 32) are constants, simplifying the solution process.
- Homogeneity: The right-hand side is zero, indicating a homogeneous differential equation, which generally has solutions involving exponential functions.

Formulating the Characteristic Equation

To solve the differential equation, the standard method involves assuming a solution of the form:

$$ y(x) = e^{rx} $$

where r is a constant to be determined. Substituting into the differential equation gives:

$$e^{rx} + 2 r e^{rx} + 16 r^2 e^{rx} + 32 r^3 e^{rx} = 0$$

Dividing through by e^{rx} (which is never zero), we obtain the characteristic equation:

$$1 + 2r + 16 r^2 + 32 r^3 = 0$$

or, written in standard polynomial form:

$$32 r^3 + 16 r^2 + 2 r + 1 = 0$$

This cubic polynomial in r is the core to finding the general solution.

Solving the Characteristic Equation

The next key step involves solving:

$$32 r^3 + 16 r^2 + 2 r + 1 = 0$$

Approach:

- Try rational root theorem: potential roots are factors of the constant term over factors of the leading coefficient, i.e., $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm \frac{1}{16}, \pm \frac{1}{32}$.

Testing these:

- For $r = -\frac{1}{2}$:

$$32 \left(-\frac{1}{2}\right)^3 + 16 \left(-\frac{1}{2}\right)^2 + 2 \left(-\frac{1}{2}\right) + 1$$

$$= 32 \left(-\frac{1}{8}\right) + 16 \left(\frac{1}{4}\right) - 1 + 1$$

$$= -4 + 4 - 1 + 1 = 0$$

Since substituting $r = -\frac{1}{2}$ yields zero, it is a root.

Factorization:

Dividing the cubic polynomial by $(r + \frac{1}{2})$:

Using synthetic division or polynomial division, we find:

$$(r + \frac{1}{2})(32 r^2 + 0 r + 2) = 0$$

Now, solving the quadratic:

$$32 r^2 + 2 = 0 \Rightarrow r^2 = - \frac{2}{32} = - \frac{1}{16}$$

$$r = \pm i \frac{1}{4}$$

Summary of roots:

- $r_1 = -\frac{1}{2}$
- $r_2 = i \frac{1}{4}$
- $r_3 = - i \frac{1}{4}$

Constructing the General Solution

The roots of the characteristic equation determine the form of the general solution:

- Real roots lead to exponential functions.
- Complex conjugate roots lead to sinusoidal functions multiplied by exponentials.

Given the roots:

- \(r = -\frac{1}{2} \) (real, negative root),
- \(r = \pm i \frac{1}{4} \) (complex conjugates).

The general solution for a third-order linear differential equation with these roots is:

\[

y(x) = C_1 e^{r_1 x} + e^{\alpha x} \left(C_2 \cos \beta x + C_3 \sin \beta x \right)

\]

where:

- \(r_1 = -\frac{1}{2} \),
- \(\alpha = 0 \) (since the imaginary roots are purely imaginary),
- \(\beta = \frac{1}{4} \).

Thus, the general solution becomes:

\[

\boxed{

y(x) = C_1 e^{-\frac{1}{2} x} + C_2 \cos \left(\frac{1}{4} x \right) + C_3 \sin \left(\frac{1}{4} x \right)

}

\]

where C_1, C_2, C_3 are arbitrary constants determined by initial conditions.

Features and Significance of the Solution

The general solution combines exponential decay and oscillatory behavior, characteristic of systems with damping and periodic components.

Features:

- Exponential decay: $e^{-\frac{1}{2} x}$ indicates damping or attenuation over x.
- Oscillatory component: $\cos (\frac{1}{4} x)$ and $\sin (\frac{1}{4} x)$ represent oscillations with a frequency of $\frac{1}{4}$.
- Linearity: The solution is a superposition of fundamental solutions, reflecting the linearity of the differential equation.

Implications:

- The presence of damping (exponential decay) suggests the system tends toward equilibrium over time.
- The oscillatory terms imply periodic behavior, useful in modeling vibrations or alternating currents.

Applications and Context

This method of solving higher-order differential equations via characteristic equations is vital in many scientific and engineering fields:

- Mechanical systems: Vibrations of beams, plates, or mass-spring systems.
- Electrical circuits: RLC circuits with multiple energy storage elements.
- Control systems: Analyzing stability and response of dynamic systems.
- Physics: Wave propagation, quantum mechanics, and oscillatory phenomena.

Understanding the solutions' structure provides insights into system behavior, stability, and response characteristics.

Pros and Cons of the Method

Pros:

- Straightforward for equations with constant coefficients.
- Systematic approach leading to explicit solutions.
- Roots of the characteristic equation directly inform the solution form.
- Well-understood mathematical framework.

Cons:

- Not applicable to equations with variable coefficients.
- Complex roots require interpretation and handling of oscillatory solutions.
- Higher-degree polynomials may be challenging to factor without computational tools.
- Initial condition incorporation might become complicated for non-homogeneous equations.

Conclusion

The problem "Find the general solution of the differential equation \(y + 2 y' + 16 y'' + 32 y''' = 0 \)" demonstrates the power and elegance of solving linear differential equations with constant coefficients. By deriving the characteristic equation, identifying roots, and constructing the general solution, we gain a comprehensive understanding of the system's behavior. This process emphasizes the importance of polynomial factorization, understanding complex roots, and the superposition principle. Such techniques are foundational in solving many real-world problems involving dynamic systems, oscillations, and damping phenomena. Mastery of these methods equips students and practitioners with essential tools for analyzing complex systems across scientific disciplines.

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