# Find The Matrix A Of The Linear Transformation T From R2 To R2 That Rotates Any Vector Through An Angle

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Understanding the behavior of linear transformations in the plane is fundamental in various fields such as computer graphics, physics, engineering, and mathematics. One of the most essential transformations in the plane is rotation, which revolves vectors around a fixed point—typically the origin—by a specified angle. This article provides a comprehensive guide to finding the matrix $A$ associated with a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that performs such rotations.

By the end of this article, you will understand how to derive the matrix representation of a rotation transformation, interpret its properties, and apply it to practical problems involving vector rotations.

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## Understanding Linear Transformations in $\mathbb{R}^2$

A linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a function that satisfies the following properties for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and scalar $c \in \mathbb{R}$:

- Additivity: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- Homogeneity: $T(c\mathbf{u}) = cT(\mathbf{u})$

Such transformations can be represented via matrices. If we fix a basis (typically the standard basis), then each linear transformation corresponds to a unique $2 \times 2$ matrix $A$, such that for any vector $\mathbf{x} \in \mathbb{R}^2$,

$$
T(\mathbf{x}) = A \mathbf{x}
$$

The core task here is to determine this matrix $A$ when $T$ is a rotation by a specific angle $\theta$.

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# What Is a Rotation in $\mathbb{R}^2$?

A rotation in the plane is a transformation that turns every vector through a fixed angle $\theta$ about the origin, without changing its magnitude.

Key properties of a rotation:

- Preserves lengths (norms) of vectors.
- Preserves angles between vectors.
- Is an orthogonal transformation (the matrix is orthogonal with determinant 1).

Mathematically, rotating a vector $\mathbf{x} = (x_1, x_2)$ by an angle $\theta$ results in a new vector $\mathbf{y} = T(\mathbf{x})$.

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# Deriving the Rotation Matrix $A$

To find the matrix $A$ corresponding to rotation by $\theta$, consider how the transformation acts on the standard basis vectors:

- $\mathbf{e}_1 = (1, 0)$
- $\mathbf{e}_2 = (0, 1)$

Since the transformation is linear, its effect on these basis vectors completely determines it.

Step 1: Rotate $\mathbf{e}_1$

Applying $T$ to $\mathbf{e}_1$:

$$
T(\mathbf{e}_1) = \text{rotation of } (1, 0) \text{ by } \theta
$$

Geometrically, rotating $\mathbf{e}_1$ by $\theta$ results in:

$$
T(\mathbf{e}_1) = (\cos \theta, \sin \theta)
$$

Step 2: Rotate $\mathbf{e}_2$

Similarly, rotating $\mathbf{e}_2 = (0, 1)$:

$$
T(\mathbf{e}_2) = (-\sin \theta, \cos \theta)
$$

Step 3: Construct the Matrix $A$

The columns of the matrix $A$ are given by the images of the basis vectors:

$$
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix, known as the rotation matrix, performs a rotation of any vector in $\mathbb{R}^2$ by the angle $\theta$.

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# Properties of the Rotation Matrix

Understanding the properties of the rotation matrix $A$ helps in analyzing its behavior and applications.

1. Orthogonality

The matrix $A$ is orthogonal:

$$
A^T A = AA^T = I
$$

which means $A^{-1} = A^T$.

2. Determinant

The determinant of $A$ is:

$$
\det(A) = \cos^2 \theta + \sin^2 \theta = 1
$$

indicating that the transformation preserves area and orientation.

3. Eigenvalues and Eigenvectors

- Eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$, complex conjugates.
- No real eigenvectors exist unless $\theta$ is a multiple of $\pi$.

4. Special Cases

- $\theta = 0$: $A = I$, identity matrix.

- $\theta = \pi$: $A = -I$, 180-degree rotation.

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# Examples of Rotation Matrices

Let's explore specific examples to solidify understanding.

Example 1: Rotation by 45 Degrees ($\pi/4$ radians)

$$
A = \begin{bmatrix}
\cos \pi/4 & -\sin \pi/4 \\
\sin \pi/4 & \cos \pi/4
\end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
$$

Example 2: Rotation by 90 Degrees ($\pi/2$ radians)

$$
A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
$$

This matrix swaps the axes with a sign change, rotating vectors by 90 degrees counterclockwise.

---

# Application of the Rotation Matrix

The rotation matrix $A$ can be used to rotate any vector $\mathbf{x} = (x_1, x_2)$:

$$
\mathbf{y} = A \mathbf{x} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}

\]

which results in:

$$
\begin{cases}
y_1 = x_1 \cos \theta - x_2 \sin \theta \\
y_2 = x_1 \sin \theta + x_2 \cos \theta
\end{cases}
$$

This is the standard rotation formula in the plane.

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# How to Find the Rotation Matrix $A$ for a Given Angle $\theta$

The process is straightforward:

1. Identify the angle $\theta$: Ensure the angle is in radians for consistency.
2. Calculate $\cos \theta$ and $\sin \theta$.
3. Construct the matrix:

$$
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

Practical steps:

- Use a calculator or software (e.g., Python, MATLAB) to compute $\cos \theta$ and $\sin \theta$.
- Plug these values into the matrix form.

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# Special Considerations and Variations

While the standard rotation matrix rotates vectors counterclockwise, other variations exist:

- Clockwise rotation: replace $\theta$ with $-\theta$:

\[
A_{cw} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\]

- Rotation by an angle about a point other than the origin: involves translation, which is not a linear transformation, but can be handled using affine transformations.

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# Applications of Rotation Matrices in Real-World Scenarios

Rotation matrices are omnipresent in numerous fields:

- Computer Graphics: rotating images and objects.
- Robotics: controlling joint angles and orientations.
- Physics: analyzing rotational motion.
- Navigation and Geospatial Coordinates: transforming coordinate systems.
- Engineering: stress analysis and structural rotations.

Understanding how to derive and apply these matrices enables professionals to perform precise transformations and simulations.

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# Summary

- A rotation in \(\mathbb{R}^2\) by an angle \(\theta\) is represented by the matrix:

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta

# Frequently Asked Questions

## What is the general form of the matrix A for a rotation transformation in R2?

The matrix A for a rotation in R2 through an angle θ is given by [[cos θ, -sin θ], [sin θ, cos θ]].

## How do you derive the matrix A for rotating vectors by an angle θ in R2?

You can derive the matrix by applying the rotation to the standard basis vectors or by using the rotation formula, resulting in A = [[cos θ, -sin θ], [sin θ, cos θ]].

## What properties does the rotation matrix A have in R2?

The rotation matrix A is orthogonal, with a determinant of 1, and preserves the length and angle of vectors.

## How does the matrix A change if the rotation angle θ is negative?

If θ is negative, the rotation matrix becomes [[cos θ, sin θ], [-sin θ, cos θ]], representing a rotation in the opposite direction.

## Can the rotation matrix A be used to rotate vectors in 3D space?

No, the matrix A described is specifically for 2D rotations; 3D rotations require 3x3 matrices and involve rotation axes.

## What is the effect of the matrix A on the length of vectors in R2?

Since A is an orthogonal matrix with determinant 1, it preserves the length of vectors, performing a pure rotation without scaling.

## How can you verify that a given matrix A represents a rotation in R2?

Verify that A is orthogonal (A^T A = I), has a determinant of 1, and that it transforms basis vectors according to the rotation angle θ.

# Additional Resources

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Introduction: The Significance of Rotation Matrices in Linear Algebra

In the realm of linear algebra, transformations serve as fundamental tools that allow mathematicians and scientists to manipulate and analyze vectors within various spaces. Among these transformations, rotations hold a central position due to their geometric and practical significance. Whether in computer graphics, robotics, physics, or engineering, understanding how to represent and compute rotations efficiently is crucial. The core mathematical object encapsulating such rotations in two-dimensional space is the rotation matrix.

This article explores the process of deriving the matrix $A$ of a linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ that performs a rotation through a specified angle. In doing so, we will analyze the geometric foundations, algebraic derivations, and practical implications of rotation matrices, providing a comprehensive understanding that bridges theory and application.

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## 1. The Geometric Intuition Behind Rotation in $\mathbb{R}^2$

### 1.1 What Does It Mean to Rotate a Vector?

A rotation in $\mathbb{R}^2$ is a transformation that turns every vector around the origin by a certain angle $\theta$, without altering its length. Geometrically, if you imagine a vector originating from the origin, rotating it by $\theta$ results in a new vector in the same plane, maintaining the same magnitude but changing the direction.

### 1.2 Visualizing Rotation

Suppose you have a point $P$ in the plane at coordinates $(x, y)$. Rotating this point about the origin by an angle $\theta$ results in a new point $P'$ with coordinates $(x', y')$. This transformation preserves the distance from the origin, satisfying the property of an isometry—specifically, a rigid motion.

### 1.3 The Role of the Rotation Angle

The angle $\theta$ determines how far the vector is rotated:

- If $\theta > 0$, the rotation is counterclockwise.
- If $\theta < 0$, the rotation is clockwise.

Understanding how to algebraically represent this geometric operation allows us to apply it systematically to any vector in $\mathbb{R}^2$.

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## 2. Mathematical Foundations of Rotation Matrices

### 2.1 Basic Principles of Linear Transformations

A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a $2 \times 2$ matrix $A$. For any vector $\mathbf{v} \in \mathbb{R}^2$, the transformation is:

$$
T(\mathbf{v}) = A \mathbf{v}
$$

The goal is to find the matrix $A$ that corresponds to a rotation through an angle $\theta$.

2.2 Properties of Rotation Matrices

Rotation matrices are characterized by the following properties:

- Orthogonality: $A^T A = I$, where $A^T$ is the transpose of $A$ and $I$ is the identity matrix.
- Determinant: $\det(A) = 1$, indicating preservation of area and orientation.
- Preservation of Lengths and Angles: Rotations are isometries.

2.3 Deriving the Rotation Matrix

The key to deriving $A$ lies in understanding how a basis vector transforms under rotation:

- The standard basis vectors are $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.
- Under a rotation by $\theta$:

$$
T(\mathbf{e}_1) = (\cos \theta, \sin \theta)
$$
$$
T(\mathbf{e}_2) = (-\sin \theta, \cos \theta)
$$

Storing these images as columns in the matrix $A$ yields:

$$
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix applies a rotation by $\theta$ to any vector $\mathbf{v}$ in $\mathbb{R}^2$.

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3. Formal Construction of the Rotation Matrix

3.1 How the Matrix Acts on a General Vector

Given an arbitrary vector $\mathbf{v} = (x, y)$, the rotation transformation is:

$$
\mathbf{v'} = A \mathbf{v} =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
$$

which expands to:

$$
\begin{cases}
x' = x \cos \theta - y \sin \theta \\
y' = x \sin \theta + y \cos \theta
\end{cases}
$$

This formula confirms that the transformation preserves the length of vectors:

$$
\| \mathbf{v'} \|^2 = x'^2 + y'^2 = (x^2 + y^2)(\cos^2 \theta + \sin^2 \theta) = x^2 + y^2
$$

since $\cos^2 \theta + \sin^2 \theta = 1$, demonstrating the isometric property of rotations.

3.2 Matrix Components and Their Geometric Interpretation

- $\cos \theta$: Projects the original vector onto the axis aligned with the original position.
- $-\sin \theta$: Represents the component perpendicular to the original projection, rotated accordingly.
- $\sin \theta$ and $\cos \theta$ in the second row: Ensure the rotation's orientation and preserve the right-handed coordinate system.

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4. Practical Application: Computing the Rotation Matrix for a Given Angle

4.1 Step-by-Step Process

To determine the matrix $A$ for a specific rotation angle $\theta$:

1. Convert $\theta$ to radians if it is given in degrees.
2. Calculate $\cos \theta$ and $\sin \theta$.
3. Construct the matrix:

$$
A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

### 4.2 Example: Rotation by 45 Degrees

Suppose we want to rotate vectors by 45° ($\pi/4$ radians):

$$
\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}
$$

Thus,

$$
A = \begin{bmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix}
$$

This matrix, when multiplied by any vector, rotates it counterclockwise by 45°.

### 4.3 Verifying the Rotation

Applying $A$ to a vector $\mathbf{v} = (x, y)$:

$$
\mathbf{v'} = \left( x \frac{\sqrt{2}}{2} - y \frac{\sqrt{2}}{2}, \, x \frac{\sqrt{2}}{2} + y \frac{\sqrt{2}}{2} \right)
$$

which geometrically aligns with a 45° rotation.

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## 5. Analytical Properties and Implications of Rotation Matrices

### 5.1 Orthogonality and Preservation of Inner Product

The matrix $A$ is orthogonal:

$$
A^T A = I
$$

which implies it preserves the dot product:

$$
\mathbf{u} \cdot \mathbf{v} = (A \mathbf{u}) \cdot (A \mathbf{v})
$$

This property is crucial in physics and computer graphics, where maintaining distances and angles is essential.

5.2 Determinant and Orientation

The determinant of $A$ is:

$$
\det(A) = \cos^2 \theta + \sin^2 \theta = 1
$$

indicating that the transformation preserves area and orientation.

5.3 Eigenvalues and Eigenvectors

The eigenvalues of $A$ are complex conjugates:

$$
\lambda_{1,2} = e^{\pm i \theta}
$$

which correspond to the rotational nature, as the matrix does not have real eigenvectors unless $\theta = 0$ or $\pi$.

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6. Broader Context and Applications

6.1 Rotation Matrices in Computer Graphics

In computer graphics, rotation matrices are used to animate objects,

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