Find The Open Interval(s) On Which The Curve Given By The Vector-valued Function Is Smooth. (Enter Your

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When analyzing vector-valued functions and their corresponding parametric curves, one of the fundamental questions is determining where the curve is smooth. A smooth curve, intuitively, is one that has no sharp corners, cusps, or points where the direction abruptly changes. Mathematically, this translates to identifying the intervals over which the derivative of the vector-valued function is continuous and non-zero. This article provides an in-depth exploration of the criteria for smoothness, methods to find these intervals, and practical examples to illustrate the concepts.

Understanding Vector-Valued Functions and Their Derivatives

Definition of a Vector-Valued Function

A vector-valued function $\mathbf{r}(t)$ maps a real parameter t to a vector in $\mathbb{R}^n$. For a typical curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, it is expressed as:

- $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ in $\mathbb{R}^2$, or
- $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ in $\mathbb{R}^3$.

The functions $x(t)$, $y(t)$, and $z(t)$ are usually assumed to be differentiable functions of t.

Derivative of a Vector-Valued Function

The derivative $\mathbf{r}'(t)$ describes the velocity vector of the curve at each point t. It is obtained by differentiating each component:

- $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ in $\mathbb{R}^2$, or
- $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ in $\mathbb{R}^3$.

The magnitude of $\mathbf{r}'(t)$, denoted $\|\mathbf{r}'(t)\|$, indicates the speed at which the point moves along the curve. The behavior of $\mathbf{r}'(t)$ influences the smoothness of the curve.

Criteria for Smoothness of a Curve

Definition of a Smooth Curve

A curve $\mathbf{r}(t)$ is considered smooth at a point $t = t_0$ if:

1. $\mathbf{r}(t)$ is differentiable at t_0,
2. $\mathbf{r}'(t)$ exists and is continuous near t_0, and
3. $\mathbf{r}'(t_0) \neq \mathbf{0}$ (the zero vector).

In essence, the curve has a well-defined, non-zero tangent vector at each point of the interval of interest, ensuring no sharp corners or cusps.

Implications of the Conditions

- If $\mathbf{r}'(t) = \mathbf{0}$ at some point, the tangent vector degenerates, and the curve may have a cusp or a point of non-differentiability there.
- The smoothness of the curve depends not only on the existence of derivatives but also on their continuity and the non-vanishing of the derivative vector.

Methodology for Finding the Smooth Intervals

Step 1: Compute the Derivative $\mathbf{r}'(t)$

Start by differentiating each component function of the vector-valued function. For example, if:

$$
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,
$$

then

$$
\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.
$$

Step 2: Determine the Magnitude $\|\mathbf{r}'(t)\|$

Calculate the magnitude to analyze where the derivative is non-zero:

$$
\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.
$$

For curves in $\mathbb{R}^2$, it simplifies to:

$$
\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}.
$$

Step 3: Identify Intervals Where $\|\mathbf{r}'(t)\|$ is Non-zero and Continuous

Find all intervals where:

- $\|\mathbf{r}'(t)\|$ is continuous, which generally follows from the differentiability of the component functions.

- $\|\mathbf{r}'(t)\| \neq 0$, ensuring the tangent vector does not vanish.

The union of these intervals constitutes the open intervals where the curve is smooth.

Step 4: Verify the Differentiability of $\mathbf{r}(t)$ and Continuity of $\mathbf{r}'(t)$

Confirm that component functions are differentiable and their derivatives are continuous over the intervals identified. If so, the curve is smooth there.

Examples and Applications

Example 1: A Simple Parametric Curve in $\mathbb{R}^2$

Consider the vector-valued function:

$$
\mathbf{r}(t) = \langle t^3 - 3t,\, t^2 - 2 \rangle.
$$

Let's analyze its smoothness over $\mathbb{R}$.

Step-by-step Solution:

1. Compute derivatives:

\[
x(t) = t^3 - 3t \Rightarrow x'(t) = 3t^2 - 3,
\]
\[
y(t) = t^2 - 2 \Rightarrow y'(t) = 2t.
\]

2. Calculate the magnitude:
\[
\|\mathbf{r}'(t)\| = \sqrt{(3t^2 - 3)^2 + (2t)^2}.
\]
\[
= \sqrt{9t^4 - 18t^2 + 9 + 4t^2} = \sqrt{9t^4 - 14t^2 + 9}.
\]

3. Find where \(\|\mathbf{r}'(t)\| \neq 0\):
\[
9t^4 - 14t^2 + 9 \neq 0.
\]
Set \(u = t^2\):
\[
9u^2 - 14u + 9 \neq 0.
\]
Solve the quadratic:
\[
9u^2 - 14u + 9 = 0,
\]
\[
u = \frac{14 \pm \sqrt{(-14)^2 - 4 \times 9 \times 9}}{2 \times 9} = \frac{14 \pm \sqrt{196 - 324}}{18}.
\]
Since discriminant is negative (\(196 - 324 = -128\)), there are no real roots, meaning the quadratic is always positive. Therefore:
\[
9t^4 - 14t^2 + 9 > 0 \quad \text{for all real } t.
\]
\[
\Rightarrow \|\mathbf{r}'(t)\| \neq 0 \quad \text{for all } t.
\]

4. Conclusion:
\(\mathbf{r}(t)\) is smooth over the entire real line \(\mathbb{R}\).

Example 2: A Curve with a Point of Non-Smoothness

Consider:
\[
\mathbf{r}(t) = \langle t^2,\, |t| \rangle.

\]
Let's analyze the smoothness.

Solution Steps:

1. Compute derivatives:
   \[
   x(t) = t

Frequently Asked Questions

How do I determine the open intervals where a vector-valued function is smooth?

To find the open intervals where a vector-valued function is smooth, identify the points where its derivatives exist and are continuous. Usually, this involves checking the derivatives of each component function and ensuring they are continuous on the intervals of interest.

What conditions must a vector-valued function satisfy to be considered smooth on an interval?

A vector-valued function is smooth on an interval if each of its component functions is continuously differentiable (C^1) on that interval, and the derivatives are continuous, ensuring the overall function has continuous derivatives.

How can I find the open interval(s) where a given vector function is not smooth?

Identify points where any component function is not differentiable or its derivative is discontinuous. The open intervals where all component derivatives are continuous and differentiable are the intervals where the vector function is smooth; the rest are points or intervals of non-smoothness.

Is the smoothness of a vector-valued function related to the smoothness of its individual component functions?

Yes, the smoothness of a vector-valued function depends on the smoothness of its component functions. The function is smooth on an interval if and only if each component function is smooth (continuously differentiable) on that interval.

What steps should I follow to find the open interval(s) where a given vector function is smooth?

First, differentiate each component function to find their derivatives. Then, determine where these derivatives are continuous and exist. The intersection of these intervals gives the open interval(s) where the vector function is smooth.

Additional Resources

Smoothness of a Curve Defined by a Vector-Valued Function: An In-Depth Exploration

Understanding the behavior of curves defined by vector-valued functions is fundamental in calculus and differential geometry. One key property of such curves is smoothness, which intuitively refers to the absence of sharp corners, cusps, or discontinuities in their trajectory. Determining the intervals where a curve is smooth is essential for applications ranging from physics to computer graphics. In this article, we delve into the nuanced process of identifying the open intervals on which a given vector-valued function defines a smooth curve, offering a comprehensive guide to this critical aspect of vector calculus.

Foundations of Smoothness in Vector-Valued Functions

What Is a Vector-Valued Function?

A vector-valued function r(t) maps a real parameter t into a vector in $\mathbb{R}^n$. Typically, in three-dimensional space, it can be expressed as:

$$
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
$$

where $x(t), y(t), z(t)$ are real-valued functions of the parameter t.

This function traces a curve in space as t varies over some interval or subset of $\mathbb{R}$. The smoothness of this curve depends on the differentiability properties of the component functions and the behavior of their derivatives.

Understanding Smoothness: The Core Concepts

Definition of a Smooth Curve

A curve $\mathbf{r}(t)$ is called smooth on an interval I if:

1. Each component function $x(t), y(t), z(t)$ is sufficiently differentiable (typically at least once).
2. The derivative $\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$ does not vanish on I, i.e., $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Key Point: The non-vanishing of the derivative ensures that the curve has a well-defined tangent vector at every point in I, and thus, the curve does not have sudden corners or cusps.

Role of the Derivative in Smoothness

The derivative $\mathbf{r}'(t)$ encapsulates the velocity vector of the moving point along the curve. If at some point t_0, $\mathbf{r}'(t_0) = \mathbf{0}$, the curve may have a point where the direction of motion is undefined or changes abruptly, indicating a potential cusp or corner.

Therefore, to find the intervals of smoothness, the primary task is to analyze where $\mathbf{r}'(t) \neq \mathbf{0}$.

Step-by-Step Approach to Find Open Intervals of Smoothness

Step 1: Compute the Derivative $\mathbf{r}'(t)$

Given the vector-valued function:

$$
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
$$

calculate:

$$
\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle
$$

This derivative exists wherever the component functions are differentiable.

Step 2: Find Where $\mathbf{r}'(t) \neq \mathbf{0}$

This involves solving:

$$
\mathbf{r}'(t) = \mathbf{0} \implies x'(t) = 0, \quad y'(t) = 0, \quad z'(t) = 0
$$

for the parameter t.

- The set of points where all derivatives are zero indicates potential singularities or non-smooth points.
- The complement of this set within the domain of $\mathbf{r}(t)$ consists of the points where the derivative is non-zero.

Note: For functions with fewer components (e.g., $\mathbb{R}^2$), the condition reduces to $x'(t) \neq 0$ or $y'(t) \neq 0$.

Step 3: Determine the Domains where $\mathbf{r}'(t)$ is Non-zero

Once the zeros of $\mathbf{r}'(t)$ are identified, the real line (or the domain of t) can be partitioned into open intervals:

- Intervals where $\mathbf{r}'(t) \neq \mathbf{0}$
- Points where $\mathbf{r}'(t) = \mathbf{0}$

The open intervals where $\mathbf{r}'(t) \neq \mathbf{0}$ are precisely the maximal open intervals of smoothness.

Practical Examples and Illustrations

Example 1: A Simple 2D Curve

Suppose:

$$
\mathbf{r}(t) = \langle t^3, t^2 \rangle
$$

Calculate:

$$
\mathbf{r}'(t) = \langle 3t^2, 2t \rangle
$$

Set $\mathbf{r}'(t) = \mathbf{0}$:

$$
3t^2 = 0 \Rightarrow t=0, \quad 2t=0 \Rightarrow t=0
$$

So, at $t=0$, the derivative vanishes, but for $t \neq 0$:

$$
\mathbf{r}'(t) \neq \mathbf{0}
$$

Intervals of smoothness:

- $\boxed{(-\infty, 0)}$
- $\boxed{(0, \infty)}$

The curve is smooth on these open intervals. At $t=0$, the derivative is zero, indicating a potential cusp or point of non-smoothness.

Example 2: A 3D Curve with a Cusp

Consider:

$$
\mathbf{r}(t) = \langle t^2, t^3, t \rangle
$$

Derivative:

$$
\mathbf{r}'(t) = \langle 2t, 3t^2, 1 \rangle
$$

Set equal to zero:

$$
2t=0 \Rightarrow t=0, \quad 3t^2=0 \Rightarrow t=0
$$

At $t=0$, the derivative:

$$
\mathbf{r}'(0) = \langle 0, 0, 1 \rangle \neq \mathbf{0}
$$

Thus, $\mathbf{r}'(t)$ is non-zero everywhere, indicating the entire real line is an interval of smoothness, despite the curve having a potential cusp in the component functions.

Special Considerations and Additional Checks

Higher-Order Derivatives and Curvature

While the primary condition for smoothness is $\mathbf{r}'(t) \neq \mathbf{0}$, in some contexts, it’s beneficial to consider higher derivatives:

- Second derivative $\mathbf{r}''(t)$ provides curvature information.
- For a curve to be regular and smooth, the tangent vector should not only be non-zero but should also vary smoothly.

Ensuring Differentiability of Component Functions

The functions $x(t), y(t), z(t)$ must be differentiable (or at least continuously differentiable) on the intervals considered. If any component functions are only piecewise differentiable, the smoothness analysis must be adapted accordingly.

Domain Restrictions

Sometimes, the domain of the vector function is restricted inherently (e.g., due to the domain of component functions). Always verify:

- The derivative exists throughout the interval.
- The derivative does not vanish within the interval.

Summary: The Procedure to Find Open Intervals of Smoothness

1. Compute $\mathbf{r}'(t)$: Find the derivatives of each component function.
2. Identify zeros of $\mathbf{r}'(t)$: Solve for t where $\mathbf{r}'(t) = \mathbf{0}$.
3. Partition the domain: Divide the domain into maximal open intervals where $\mathbf{r}'(t) \neq \mathbf{0}$.
4. Confirm smoothness: On these intervals, confirm that the component functions are differentiable and $\mathbf{r}'(t) \neq \mathbf{0}$.
5. State the open intervals: These are the regions where the curve is smooth.

Conclusion: The Significance of Identifying Open Intervals of Smoothness

Determining the open intervals where a vector-valued function defines a smooth curve is not merely an academic exercise—it is crucial for understanding the geometry and behavior of

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