Find The Z-transfer Function H(z) Of The Following Discrete-time Syst Ifference Equation: Y(n) = Y(n-1)

Find The Z-transfer Function H(z) Of The Following Discrete-time System Difference Equation: Y(n) = Y(n-1)

Understanding the Z-transform and its application to discrete-time systems is fundamental in control theory, signal processing, and digital system analysis. In this article, we focus on deriving the Z-transform transfer function, H(z), for a specific difference equation: Y(n) = Y(n-1). This simple yet significant relation forms the basis for understanding more complex recursive systems and their behaviors. Through detailed explanations, step-by-step derivations, and practical insights, we aim to provide a comprehensive guide to analyzing this system and calculating its transfer function.

Introduction to Discrete-Time Systems and Difference Equations

What Are Discrete-Time Systems?

Discrete-time systems process signals that are defined only at discrete time intervals, typically represented by integer indices n. These systems are prevalent in digital signal processing, digital control systems, and computer-based applications. They are characterized by their input-output relationships, often modeled through difference equations.

Difference Equations in Discrete-Time Systems

Difference equations describe the relationship between current and past values of signals. They are the discrete counterpart of differential equations in continuous systems. For example, the difference equation:

\[Y(n) = a \, Y(n-1) + b \, X(n) \]

links the current output to previous outputs and current inputs, with coefficients a and b defining the system's properties.

Understanding the Given Difference Equation: Y(n) = Y(n-1)

System Description

The difference equation under consideration:

$$ Y(n) = Y(n-1) $$

is a homogeneous, recursive relation. It states that the current output Y(n) is equal to the previous output Y(n-1), without any external input influence.

Implications of the Equation

- No Input Term: The system is autonomous; it has no external input (or input is zero).
- Memoryless Nature: The output depends solely on its past value.
- Stability and Behavior: Depending on initial conditions, the system's output will either remain constant or diverge.

Applying the Z-transform to the Difference Equation

What is the Z-transform?

The Z-transform converts discrete-time signals from the time domain into the complex frequency domain, facilitating the analysis of system properties like stability, frequency response, and transfer functions.

The Z-transform of a sequence $y(n)$ is defined as:

$$ Y(z) = \sum_{n=0}^{\infty} y(n) z^{-n} $$

where z is a complex variable.

Transforming the Difference Equation

Applying the Z-transform to both sides of the difference equation:

$$ Y(n) = Y(n-1) $$

we obtain:

$$ Z\{Y(n)\} = Z\{Y(n-1)\} $$

which simplifies to:

$$ Y(z) = z^{-1} Y(z) + y(-1) z^{0} $$

However, since the sequence is causal and typically defined for $n \geq 0$, we assume initial conditions $y(-1) = 0$ for simplicity, unless specified otherwise.

Thus, the Z-transform of the difference equation becomes:

$$ Y(z) = z^{-1} Y(z) $$

Note: For the initial condition $y(-1) = 0$, the above holds. If initial conditions are non-zero, they need to be incorporated.

Deriving the Z-Transfer Function H(z)

Definition of Transfer Function in Z-domain

The Z-transfer function $H(z)$ relates the output $Y(z)$ to the input $X(z)$:

$$H(z) = \frac{Y(z)}{X(z)}$$

In systems where the input is zero or absent, $H(z)$ characterizes the system's inherent behavior.

Analyzing the System Without Input

Since the difference equation contains no input term, this system is purely recursive with no external excitation. For such systems, the transfer function $H(z)$ is generally derived in the context of an input-output relation. To proceed, suppose an input $X(n)$ is applied, and the system is described by:

$$Y(n) = Y(n-1) + b \, X(n)$$

but in our case, the equation is:

$$Y(n) = Y(n-1)$$

implying no input-driven change.

In the absence of an input, the transfer function is less meaningful as a ratio of output to input. However, if we interpret the equation as part of a system with input $X(n)$ where the output is fed back, the transfer function can be derived accordingly.

Alternatively, consider the homogeneous system:

$$Y(n) = Y(n-1)$$

which has characteristic solutions.

Solution of the Difference Equation and System Behavior

General Solution

The difference equation:

$$Y(n) = Y(n-1)$$

has the general solution:

$$Y(n) = C$$

where C is a constant determined by initial conditions. This indicates that the output remains constant over time.

If the initial condition $Y(0) = Y_0$, then:

$$Y(n) = Y_0$$

This steady-state solution implies that the system is a memory element maintaining its initial state indefinitely.

Stability Analysis

The characteristic equation:

$$r - 1 = 0$$

has a root:

$$r = 1$$

Since the root magnitude is 1, the system is marginally stable. It neither decays nor grows, implying a persistent output determined by initial conditions.

Deriving the Transfer Function H(z) for the System

Considering a More General System

To derive a meaningful $H(z)$, we often consider systems where the output depends on previous outputs and inputs, such as:

$$Y(n) - a \, Y(n-1) = b \, X(n)$$

which leads to the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - a z^{-1}}$$

In our case, the difference equation:

$$Y(n) = Y(n-1)$$

can be viewed as a special case with $a = 1$ and $b = 0$, implying:

$$Y(n) - 1 \times Y(n-1) = 0$$

and thus:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0}{1 - z^{-1}} = 0$$

which indicates no input influence.

Transfer Function of the Autonomous System

Given the autonomous nature of the system, the transfer function $H(z)$ is generally considered as:

$$H(z) = \frac{Y(z)}{Y(z)} = 1$$

but only in the context of initial conditions and internal feedback.

Therefore, the key understanding is:

- The system acts as a memory element holding its initial value.
- The transfer function is unity when considering internal feedback or initial state.

Practical Implications and Applications

System Behavior in Real-World Scenarios

- Memory Storage: The difference equation models a system that maintains its state, useful in digital memory elements.
- Stability: The root at $r = 1$ indicates the system's marginal stability, meaning it neither amplifies nor diminishes signals over time.

Design Considerations

- Initial Conditions: The initial state $Y(0)$ heavily influences the system's output.
- External Inputs: For systems with inputs, the transfer function relates inputs to outputs and guides filter design.

Summary and Key Takeaways

1. **The difference equation $Y(n) = Y(n-1)$ describes a system with persistent state.**

2. **Applying the Z-transform yields the relation $Y(z) = z^{-1} Y(z)$, leading to insights about system behavior.**

3. **The transfer function $H(z)$ in the autonomous case is effectively unity or undefined in the**

traditional sense, emphasizing internal memory rather than input-output dynamics.

4. The system's characteristic root at $r=1$ indicates marginal stability and dependence on initial conditions.

5. Understanding such fundamental difference equations is crucial for analyzing more complex recursive digital systems and designing stable, reliable digital filters.

Conclusion

The difference equation $Y(n) = Y(n-1)$ represents a fundamental example in discrete-time system analysis. While it lacks an explicit input, analyzing its Z-transform and inherent transfer function provides valuable insights into system stability, memory, and behavior. Recognizing the implications of such simple recursive relations forms the foundation for understanding more sophisticated digital control

Frequently Asked Questions

What is the Z-transfer function H(z) of the discrete-time system defined by Y(n) = Y(n-1)?

The Z-transfer function H(z) for the system Y(n) = Y(n-1) is H(z) = 1/z.

How do you derive the transfer function H(z) from the difference equation Y(n) = Y(n-1)?

By taking the Z-transform of both sides, assuming zero initial conditions, the difference equation becomes $Y(z) = z^{-1}Y(z)$, which simplifies to $H(z) = Y(z)/X(z) = 1/z$ for a system with input X(n). In this specific case, since the equation involves only Y(n), H(z) is 1/z.

Is the difference equation Y(n) = Y(n-1) a stable system, and what does its transfer function indicate about stability?

The difference equation represents a system where the current output equals the previous output, which is marginally stable (it neither decays nor grows). The transfer function H(z) = 1/z indicates a pole at the origin, suggesting marginal stability, but the system's behavior depends on initial conditions.

Can the difference equation Y(n) = Y(n-1) be considered a recursive or non-recursive system?

It is a recursive system because the current output Y(n) depends on the previous output Y(n-1).

What is the significance of the pole at z=0 in the transfer function H(z) = 1/z for the difference equation Y(n) = Y(n-1)?

The pole at z=0 indicates that the system is a recursive one with a pole at the origin, which leads to marginal stability and suggests that the output may persist indefinitely without decaying.

Additional Resources

Find The Z-transfer Function H(z) Of The Following Discrete-time System Difference Equation: Y(n) = Y(n-1)

Understanding the z-transfer function of a discrete-time system is fundamental in digital signal processing and control systems. In this review, we will delve into the process of deriving the z-transfer function H(z) for the given difference equation Y(n) = Y(n-1). This simple yet instructive system provides a perfect foundation for grasping the concepts of system analysis in the z-domain, which is essential for designing digital filters, analyzing system stability, and understanding system behavior.

Introduction to Discrete-time Systems and Difference Equations

Discrete-time systems are systems where signals are defined only at discrete time intervals. They are widely used in digital signal processing, computer control systems, and communication systems. The behavior of such systems is often characterized by difference equations, which relate the current output to past outputs and inputs.

A difference equation like Y(n) = Y(n-1) is a first-order homogeneous linear difference equation with constant coefficients. It describes a system where the current output depends solely on the previous output, indicating a recursive process.

The key to analyzing such systems lies in transforming the difference equation into the z-domain, where algebraic methods can be applied more straightforwardly, enabling us to determine the system's transfer function H(z).

Understanding the Difference Equation Y(n) = Y(n-1)

This specific difference equation states that:

- The current output Y(n) is equal to the previous output Y(n-1).
- It implies a system with no input influence, which suggests the system is purely recursive with initial conditions determining its behavior.
- The equation is homogeneous, meaning if initial conditions are zero, the system's output remains zero for all time, unless external input is introduced.

From a physical standpoint, this system models a process where the output maintains its previous state indefinitely, akin to an integrator with no input, or a system with a "memory" that persists over time.

Deriving the Z-Transform of the Difference Equation

The first step in obtaining the transfer function is applying the Z-transform to the difference equation.

Recall that the Z-transform of a discrete-time sequence $Y(n)$ is defined as:

$$ Y(z) = \sum_{n=0}^{\infty} Y(n) z^{-n} $$

Applying the Z-transform to both sides of $Y(n) = Y(n-1)$:

$$ Z\{Y(n)\} = Z\{Y(n-1)\} $$

Using the property of the Z-transform for a shifted sequence:

$$ Z\{Y(n-1)\} = z^{-1} [Y(z)] + y(-1) $$

where $y(-1)$ is the initial condition. Usually, in system transfer function derivations, initial conditions are assumed to be zero (causality assumption), simplifying the analysis.

Assuming zero initial conditions:

$$ Y(z) = z^{-1} Y(z) $$

Rearranged, this becomes:

$$ Y(z) - z^{-1} Y(z) = 0 $$

or

$$ Y(z) (1 - z^{-1}) = 0 $$

which leads us to the next step: defining the transfer function.

Defining the Z-Transfer Function H(z)

The z-transfer function H(z) relates the output Y(z) to the input X(z) in the Z-domain:

$$ H(z) = \frac{Y(z)}{X(z)} $$

However, in the absence of an explicit input x(n), the system is purely recursive with initial conditions. To make the system meaningful in the context of transfer functions, we typically consider the system as a response to an input. For the current difference equation, if we interpret the system as having no input, the transfer function reduces to analyzing the system's inherent behavior.

In general, for a system described by a difference equation with input, the difference equation would look like:

$$ Y(n) + a_1 Y(n-1) + \cdots + a_N Y(n-N) = b_0 X(n) + b_1 X(n-1) + \cdots + b_M X(n-M) $$

But in our case, since no input appears, the transfer function essentially depicts the system's stability and response to initial conditions.

Deriving H(z) for the Difference Equation Y(n) = Y(n-1)

Given the previous discussion, the difference equation simplifies to:

$$ Y(n) = Y(n-1) $$

Applying the Z-transform with zero initial conditions yields:

$$ Y(z) = z^{-1} Y(z) $$

which implies:

$$ Y(z) (1 - z^{-1}) = 0 $$

or equivalently,

$$ (z - 1) Y(z) = 0 $$

This indicates that:

$$ Y(z) = 0 \quad \text{or} \quad z = 1 $$

Now, if we introduce an external input input $X(n)$ to analyze the system's transfer function, the system would be characterized as:

$$ Y(n) = Y(n-1) + X(n) $$

which is a typical first-order system with input. But since the original equation only involves $Y(n)$, the system is autonomous, and its transfer function in the traditional sense is trivial.

However, assuming the system is driven by an input and the system is defined as:

$$ Y(n) = Y(n-1) + X(n) $$

then the Z-transform leads to:

$$ Y(z) = z^{-1} Y(z) + X(z) $$

Rearranging:

$$ Y(z) (1 - z^{-1}) = X(z) $$

$$ Y(z) = \frac{X(z)}{1 - z^{-1}} $$

Multiplying numerator and denominator by z:

$$ Y(z) = \frac{z X(z)}{z - 1} $$

Thus, the transfer function becomes:

$$ H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - 1} $$

Features and Analysis of the Transfer Function H(z) = z / (z - 1)

This transfer function characterizes a system with specific stability and frequency response characteristics.

Features:

- Pole Location: The pole is at $z = 1$, which lies on the unit circle, indicating marginal stability (neither strictly stable nor unstable).
- Zero Location: There's a zero at the origin ($z = 0$), affecting the system's frequency response, especially at high frequencies.
- System Type: First-order, recursive system with a simple pole-zero structure.
- Frequency Response: As $z \to e^{j\omega}$, the magnitude of H(z) approaches infinity at $\omega = 0$, indicating a low-frequency gain boost.

Pros:

- Simple analytical form allows easy analysis of stability and frequency response.
- Useful for understanding basic recursive filter behavior.

- Serves as a building block for more complex systems.

Cons:

- Marginal stability at $z=1$ can lead to unbounded outputs if not managed properly.
- The system's response to initial conditions needs careful handling.
- Not directly applicable unless an input is considered, as the original difference equation is autonomous.

Implications in System Design and Signal Processing

Understanding H(z) = z / (z - 1) has practical implications:

- Filter Design: This transfer function resembles that of an integrator in digital systems, which is fundamental in designing accumulative filters or integrators.
- Stability Considerations: The pole on the unit circle requires careful handling to prevent unbounded output, especially in real-world implementations.
- System Behavior: The system exhibits a tendency to accumulate past inputs, leading to potential issues like drift or saturation in practical applications.

Summary and Final Thoughts

The process of deriving the z-transfer function for the difference equation Y(n) = Y(n-1) reveals key insights into the nature of recursive systems. Assuming the system is driven by an input, the transfer function simplifies to H(z) = z / (z - 1), embodying a first-order recursive system with a pole on the unit circle. Such systems are fundamental in understanding digital filters and recursive algorithms.

In conclusion, while the original difference equation suggests an autonomous system, extending it with an input component provides a meaningful transfer function that encapsulates the system's dynamics. Recognizing the location of poles and zeros in the z-plane offers invaluable information regarding system stability, frequency response, and potential practical issues in implementation.

This foundational analysis paves the way for more advanced studies involving higher-order systems, stability criteria, and filter design methods, making it an essential part of the signal processing toolkit.

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