For The Following Second-order System And Initial Conditions, Find The Transient Solution: X + 88 + 12X

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Understanding how to analyze and solve second-order systems is fundamental in fields like control engineering, mechanical vibrations, and electrical circuit design. When given a second-order differential equation with specific initial conditions, finding the transient response helps engineers predict system behavior before reaching steady-state. This article provides a comprehensive, step-by-step guide to solving such a system, focusing on the process of deriving the transient solution, with detailed explanations and practical insights.

Introduction to Second-Order Systems

Second-order systems are characterized by differential equations of the form:

$$ a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + c x = f(t) $$

where a, b, and c are constants, and $f(t)$ is the forcing function. These systems exhibit dynamic behaviors such as oscillations, damping, and overshoot, which are crucial in designing stable and responsive systems.

Key points about second-order systems:

- They can be underdamped, overdamped, or critically damped based on system parameters.
- The transient response describes the system's behavior before reaching steady-state.
- The steady-state response depends on the input, while the transient response is influenced by initial conditions.

Understanding the Given System

The presented problem involves the differential equation:

$$ X + 88 + 12X $$

However, this expression appears incomplete or incorrectly formatted as a differential equation. Assuming the intended form is a standard second-order differential equation, a typical example might be:

$$ \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 12x = 0 $$

or, if the coefficients differ, a similar second-order form.

Note: For clarity, we'll interpret the problem as solving the homogeneous second-order differential equation:

$$ \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 12x = 0 $$

with given initial conditions:

$$ x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}\bigg|_{t=0} = v_0 $$

which are typically provided in such problems.

Initial conditions are essential for determining the particular solution of the differential equation and thus finding the transient response.

Step-by-Step Solution Approach

To find the transient solution of a second-order system, follow these key steps:

1. Write the Differential Equation

Identify the standard form of the second-order differential equation based on system parameters. For example:

$$ \frac{d^2x}{dt^2} + b \frac{dx}{dt} + c x = 0 $$

2. Find the Characteristic Equation

Convert the differential equation into its characteristic form:

$$ r^2 + b r + c = 0 $$

Solve for the roots r_1 and r_2:

- If roots are real and distinct: overdamped system
- If roots are real and repeated: critically damped system
- If roots are complex conjugates: underdamped system

3. Determine the Homogeneous Solution

Based on the roots:

- Real and distinct roots:

$$ x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} $$

- Repeated roots:

$$ x_h(t) = (A + B t) e^{r t} $$

- Complex conjugate roots:

$$ x_h(t) = e^{\alpha t} (D \cos \beta t + E \sin \beta t) $$

where $r = \alpha \pm j \beta$.

4. Apply Initial Conditions to Find Constants

Use initial position $x(0) = x_0$ and initial velocity $\frac{dx}{dt}\big|_{t=0} = v_0$ to solve for constants C_1 and C_2 (or A, B, D, E).

5. Write the Transient Solution

The transient solution is the homogeneous part of the total solution, which describes how the system responds over time before reaching steady-state.

Example: Solving a Second-Order System with Given Initial Conditions

Let's consider a specific example:

$$ \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 12x = 0 $$

with initial conditions:

$$ x(0) = 5 \quad \text{and} \quad \frac{dx}{dt}\bigg|_{t=0} = 0 $$

Step 1: Write the Characteristic Equation

$$ r^2 + 8 r + 12 = 0 $$

Step 2: Solve for Roots

$$ r = \frac{-8 \pm \sqrt{64 - 48}}{2} = \frac{-8 \pm \sqrt{16}}{2} $$

$$ r = \frac{-8 \pm 4}{2} $$

- $r_1 = \frac{-8 + 4}{2} = -2$
- $r_2 = \frac{-8 - 4}{2} = -6$

Step 3: Write General Homogeneous Solution

Since roots are real and distinct:

$$ x_h(t) = C_1 e^{-2 t} + C_2 e^{-6 t} $$

Step 4: Find Derivative and Apply Initial Conditions

$$ \frac{dx}{dt} = -2 C_1 e^{-2 t} - 6 C_2 e^{-6 t} $$

At $t=0$:

$$ x(0) = C_1 + C_2 = 5 $$

$$ \frac{dx}{dt}\big|_{t=0} = -2 C_1 - 6 C_2 = 0 $$

Step 5: Solve for Constants

From the first equation:

$$ C_1 + C_2 = 5 $$

From the second:

$$ -2 C_1 - 6 C_2 = 0 $$

Multiply the first equation by 2:

$$ 2 C_1 + 2 C_2 = 10 $$

Add to the second equation:

$$ (-2 C_1 - 6 C_2) + (2 C_1 + 2 C_2) = 0 + 10 $$

$$ (-2 C_1 + 2 C_1) + (-6 C_2 + 2 C_2) = 10 $$

$$ 0 - 4 C_2 = 10 \Rightarrow C_2 = -\frac{10}{4} = -2.5 $$

Substitute into the first:

$$ C_1 - 2.5 = 5 \Rightarrow C_1 = 7.5 $$

Final Transient Solution:

$$ x(t) = 7.5 e^{-2 t} - 2.5 e^{-6 t} $$

This solution describes the transient behavior of the system based on initial conditions.

Interpreting the Transient Response

The transient response depicts how the system's output evolves over time from the initial state toward equilibrium. Key characteristics include:

- Decay rate: determined by the exponential terms ($e^{r t}$)
- Oscillations: occur if roots are complex conjugates
- Steady-state: as $t \to \infty$, the exponential terms approach zero, leaving the system at equilibrium

Practical insights:

- Faster decay rates imply quicker stabilization.
- The initial conditions influence the amplitude and phase of the transient response.
- Engineers can modify system parameters to optimize transient behavior for desired performance.

Additional Considerations

Effects of Damping

The damping coefficient (b) influences whether the system is underdamped, overdamped, or critically damped:

- Underdamped: complex roots, oscillatory transient response
- Overdamped: real, distinct roots, non-oscillatory but slow decay
- Critically damped: repeated roots, fastest non-oscillatory response

Steady-State vs. Transient Response

While the transient solution describes initial dynamics, the steady-state solution reflects long-term behavior. For homogeneous equations, the steady-state is often zero unless a forcing function is present.

Practical Applications

Understanding transient solutions is critical in:

- Designing control systems with desired response times
- Analyzing mechanical vibrations to prevent resonance
- Electrical circuit analysis for transient currents and voltages

Conclusion

Finding the transient solution of a second-order system involves understanding the differential equation, solving the characteristic equation, applying initial conditions, and interpreting the resulting response. Properly analyzing these solutions enables engineers and scientists to design more stable, efficient, and responsive systems across various fields. Whether dealing with mechanical vibrations, electrical circuits, or control systems, mastering the process of deriving transient responses is a vital skill in system dynamics analysis.

Summary of key steps:

1. Formulate the differential equation
2. Solve the characteristic equation
3. Write the general solution

Frequently Asked Questions

What is the general form of the second-order differential equation given by X + 88 + 12X?

The equation appears to be miswritten; assuming it represents a second-order differential equation, it might be intended as X'' + 88X' + 12X = 0, which is a standard form for a second-order homogeneous differential equation.

How do you interpret the initial conditions when solving for the transient response?

Initial conditions specify the initial displacement and velocity (X(0) and X'(0)) of the system, which are used to solve for the specific constants in the homogeneous solution and determine the transient response.

What is the method to find the transient solution of a second-order homogeneous differential equation?

The method involves solving the characteristic equation associated with the differential equation to find the roots, then writing the general solution based on these roots, and finally applying initial conditions to find particular constants.

How do you form the characteristic equation for the differential equation X'' + 88X' + 12X = 0?

Replace X'' with r^2, X' with r, and X with 1 to get r^2 + 88r + 12 = 0, which is the characteristic equation.

What are the steps to solve for the transient response once the characteristic roots are found?

First, solve the characteristic equation to find roots r1 and r2. Then, write the general solution as X(t) = C1 e^{r1 t} + C2 e^{r2 t}. Next, apply initial conditions to solve for C1 and C2.

How does the discriminant of the characteristic equation

influence the nature of the transient response?

The discriminant (D = 88^2 - 4112) determines whether the roots are real and distinct, real and repeated, or complex conjugates, which affects whether the response is overdamped, critically damped, or underdamped.

What is the typical form of the transient solution for complex conjugate roots?

The transient solution takes the form X(t) = e^{α t} (A cos β t + B sin β t), where α is the real part of the roots and β is the imaginary part.

How do initial conditions affect the calculation of constants in the transient solution?

Initial conditions are used to set up equations for C1 and C2 (or A and B), which are then solved to tailor the general solution to the specific initial state of the system.

What is the significance of the transient solution in the overall response of the system?

The transient solution describes the system's temporary response that decays over time, helping to understand how the system settles to steady-state behavior after initial disturbances.

Can the given equation X + 88 + 12X be directly used to find the transient solution?

No, as written, the equation appears incomplete or incorrectly formatted. A proper second-order differential equation, such as X'' + 88X' + 12X = 0, is necessary to find the transient solution.

Additional Resources

For The Following Second-order System And Initial Conditions, Find The Transient Solution

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Introduction

In the realm of control systems and differential equations, understanding the transient response of second-order systems is fundamental. These systems, characterized by their second derivative terms, often model real-world phenomena such as mechanical vibrations, electrical circuits, and dynamic systems in engineering disciplines. This article aims to thoroughly investigate the problem statement: "For the following second-order system and initial conditions, find the transient solution," specifically focusing on the differential equation:

$$ X'' + 8X' + 12X = 0 $$

with given initial conditions. We will explore the theoretical background, solution methodology, step-by-step derivation, and interpret the physical significance of the transient response.

Background on Second-Order Differential Equations

Second-order differential equations are equations involving the second derivative of a function, typically expressed as:

$$ aX'' + bX' + cX = f(t) $$

where a, b, c are constants, and $f(t)$ is the forcing function. When $f(t) = 0$, the system is said to be homogeneous, and solutions characterize the natural response of the system—the transient behavior.

The general form of the homogeneous second-order differential equation is:

$$ aX'' + bX' + cX = 0 $$

which describes systems like mass-spring-damper setups, RLC circuits, and other physical models.

Significance of the Transient Solution

The transient solution describes how the system behaves over time after an initial disturbance and before reaching steady-state. It captures oscillations, damping, and decay characteristics, which are crucial for stability analysis and system design.

Problem Statement and System Description

Given the differential equation:

$$ X'' + 8X' + 12X = 0 $$

and initial conditions:

$$ X(0) = X_0 $$
$$ X'(0) = V_0 $$

The goal is to find the transient solution $X(t)$ that satisfies both the differential equation and the initial conditions.

Step 1: Homogeneous Differential Equation Analysis

The differential equation is homogeneous and linear with constant coefficients. To solve, we first find the characteristic equation:

$$ r^2 + 8r + 12 = 0 $$

This quadratic can be solved using the quadratic formula:

$$ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1$, $b=8$, and $c=12$.

Calculating:

$$ r = \frac{-8 \pm \sqrt{64 - 48}}{2} $$
$$ r = \frac{-8 \pm \sqrt{16}}{2} $$
$$ r = \frac{-8 \pm 4}{2} $$

Thus, the roots are:

- $r_1 = \frac{-8 + 4}{2} = -2$
- $r_2 = \frac{-8 - 4}{2} = -6$

Step 2: General Solution of the Homogeneous Equation

Since the roots are real and distinct, the general solution is:

$$ X(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} $$
$$ X(t) = C_1 e^{-2t} + C_2 e^{-6t} $$

where C_1 and C_2 are constants determined by initial conditions.

Step 3: Applying Initial Conditions

To find C_1 and C_2, differentiate $X(t)$:

$$ X'(t) = -2 C_1 e^{-2t} - 6 C_2 e^{-6t} $$

At $t=0$:

$$ X(0) = C_1 + C_2 = X_0 $$
$$ X'(0) = -2 C_1 - 6 C_2 = V_0 $$

These form a system of equations:

$$
\begin{cases}
C_1 + C_2 = X_0 \\
-2 C_1 - 6 C_2 = V_0
\end{cases}
$$

Step 4: Solving for C_1 and C_2

From the first equation:

$$ C_1 = X_0 - C_2 $$

Substitute into the second:

$$ -2 (X_0 - C_2) - 6 C_2 = V_0 $$

Simplify:

$$ -2 X_0 + 2 C_2 - 6 C_2 = V_0 $$
$$ -2 X_0 - 4 C_2 = V_0 $$

Solve for C_2:

$$ -4 C_2 = V_0 + 2 X_0 $$
$$ C_2 = - \frac{V_0 + 2 X_0}{4} $$

Then:

$$ C_1 = X_0 - C_2 = X_0 + \frac{V_0 + 2 X_0}{4} = \frac{4 X_0 + V_0 + 2 X_0}{4} = \frac{6 X_0 + V_0}{4} $$

Transient Solution Expression

The complete transient solution is:

$$ \boxed{
X(t) = \frac{6 X_0 + V_0}{4} e^{-2t} - \frac{V_0 + 2 X_0}{4} e^{-6t}
} $$

This formula describes how the system's response decays over time, with initial conditions dictating the specific amplitude and phase.

Physical Interpretation of the Transient Response

The exponential terms e^{-2t} and e^{-6t} indicate damping in the system. The rates -2 and -6 correspond to the system's damping characteristics; the faster decay associated with e^{-6t} suggests a more heavily damped mode.

The coefficients C_1 and C_2 depend on the initial displacement X_0 and initial velocity V_0. For example:

- A larger initial velocity V_0 amplifies the transient oscillations.
- If initial conditions are zero, the transient response is zero, indicating the system starts at equilibrium.

Special Cases and Further Analysis

1. Zero Initial Displacement and Velocity:

$$ X_0 = 0, \quad V_0 = 0 \Rightarrow X(t) = 0 $$

The system remains at equilibrium.

2. Overdamped Behavior:

The two distinct real roots (-2 and -6) confirm overdamping, implying no oscillations but exponential decay toward equilibrium.

3. Damping Ratio and Natural Frequency:

The characteristic roots relate to the damping ratio ζ and natural frequency ω_n:

$$ r_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} $$

Since roots are real and negative, the damping ratio $\zeta > 1$, indicating an overdamped system.

Conclusion

The detailed analysis of the second-order system governed by:

$$ X'' + 8X' + 12X = 0 $$

demonstrates the derivation of the transient solution, elucidating the system's natural response characteristics. The solution's exponential decay terms highlight the overdamped nature, crucial for engineers and scientists modeling physical systems where transient behavior dominates initial responses.

By systematically applying the characteristic equation, initial condition constraints, and physical interpretation, this investigation provides a comprehensive methodology for analyzing similar second-order systems in engineering and physics.

References

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This review underscores the importance of precise mathematical techniques in understanding the transient dynamics of second-order systems, with applications spanning various engineering disciplines.

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