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When faced with a second-order linear differential equation such as $Y''' + 8Y' + 17Y = 0$, applying the Laplace Transform provides a powerful method for solving the problem, especially when initial conditions are given. This technique transforms the differential equation from the time domain into algebraic equations in the complex frequency domain, simplifying the process of finding the solution. In this article, we will explore the step-by-step application of the Laplace Transform to the differential equation, including handling initial conditions, solving the algebraic equation, and performing the inverse Laplace Transform to find $y(t)$.

Understanding the Differential Equation

The given differential equation is:

$$
Y''' + 8Y' + 17Y = 0
$$

with initial conditions:

$$
Y(0) = 0, \quad Y'(0) = 2
$$

Note: Typically, for a second-order differential equation, initial conditions include $Y(0)$ and $Y'(0)$. However, the notation suggests the equation involves third derivatives—it's important to clarify that the notation Y''' indicates the third derivative. If the original problem intended a second-order differential equation, it might have been written as $y'' + 8y' + 17y = 0$. But as per the provided equation, it is third order.

For clarity, assume the differential equation is:

$$
Y''' + 8Y' + 17Y = 0
$$

with initial conditions:

$$
Y(0) = 0, \quad Y'(0) = 2, \quad Y''(0) = ?
$$

But since only two initial conditions are provided, it suggests perhaps the original equation is second order. Alternatively, the initial conditions correspond to $Y(0)$ and $Y'(0)$ for a second-order differential equation:

$$
Y'' + 8Y' + 17Y = 0
$$

Given the context, and to align with typical differential equations of this form, let's proceed assuming the differential equation is:

$$
Y'' + 8Y' + 17Y = 0
$$

with initial conditions:

$$
Y(0) = 0, \quad Y'(0) = 2
$$

This assumption makes the problem more consistent with standard second-order ODEs.

Applying the Laplace Transform

The Laplace Transform converts derivatives into algebraic expressions involving the complex variable s:

$$
\mathcal{L}\{y(t)\} = Y(s) = \int_0^{\infty} y(t) e^{-st} dt
$$

Key properties for derivatives:

- $\mathcal{L}\{ y'(t) \} = sY(s) - y(0)$
- $\mathcal{L}\{ y''(t) \} = s^2 Y(s) - s y(0) - y'(0)$

Transforming the Differential Equation

Applying the Laplace Transform to the differential equation:

$$
Y'' + 8Y' + 17Y = 0
$$

we get:

$$
\mathcal{L}\{Y''\} + 8 \mathcal{L}\{Y'\} + 17 \mathcal{L}\{Y\} = 0
$$

which simplifies to:

$$
(s^2 Y(s) - s y(0) - y'(0)) + 8 (sY(s) - y(0)) + 17 Y(s) = 0
$$

Plugging in the initial conditions:

$$
y(0) = 0, \quad y'(0) = 2
$$

we have:

$$
(s^2 Y(s) - s \times 0 - 2) + 8 (sY(s) - 0) + 17 Y(s) = 0
$$

This simplifies to:

$$
s^2 Y(s) - 2 + 8s Y(s) + 17 Y(s) = 0
$$

Simplifying the Algebraic Equation

Group the terms with $Y(s)$:

$$
(s^2 + 8s + 17) Y(s) = 2
$$

Thus, the Laplace Transform of the solution is:

$$
Y(s) = \frac{2}{s^2 + 8s + 17}
$$

Finding the Inverse Laplace Transform

Now, the goal is to find $y(t)$ by applying the inverse Laplace Transform to:

$$
Y(s) = \frac{2}{s^2 + 8s + 17}
$$

Completing the Square in the Denominator

Rewrite the quadratic:

$$
s^2 + 8s + 17
$$

Complete the square:

$$
s^2 + 8s + 16 + 1 = (s + 4)^2 + 1
$$

So,

$$
Y(s) = \frac{2}{(s + 4)^2 + 1}
$$

Using Standard Laplace Transform Pairs

Recall the standard inverse Laplace transforms:

- $\mathcal{L}^{-1}\left\{ \frac{\omega}{(s - a)^2 + \omega^2} \right\} = e^{a t} \sin$

\omega t \)
- \(\mathcal{L}^{-1}\left\{ \frac{s - a}{(s - a)^2 + \omega^2} \right\} = e^{a t} \cos \omega t \)

In our case:

\[
Y(s) = \frac{2}{(s + 4)^2 + 1}
\]

which matches the form:

\[
\frac{\omega}{(s - a)^2 + \omega^2}
\]

with \(a = -4 \), \(\omega = 1 \), and numerator 2.

Expressed as:

\[
Y(s) = 2 \times \frac{1}{(s + 4)^2 + 1}
\]

Therefore, the inverse Laplace transform is:

\[
y(t) = 2 e^{-4 t} \sin t
\]

Final Solution

The solution to the differential equation with the given initial conditions is:

\[
\boxed{
y(t) = 2 e^{-4 t} \sin t
}
\]

Summary of the Solution Process

To summarize, solving the differential equation $y'' + 8 y' + 17 y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 2$ via the Laplace Transform involves:

1. Transforming the differential equation into an algebraic equation in the s-domain, incorporating initial conditions.
2. Solving for $Y(s)$, the Laplace Transform of the solution.
3. Completing the square in the denominator to recognize standard inverse Laplace transform forms.
4. Applying the inverse Laplace Transform to find $y(t)$ explicitly.
5. Interpreting the solution in the context of the original differential equation.

Additional Considerations

- If the original differential equation was third order, similar steps apply but with more initial conditions and a more complex algebraic expression.
- The Laplace Transform method is especially useful for linear differential equations with constant coefficients and initial value problems.
- Understanding standard Laplace pairs and properties simplifies the inverse transformation process.

Practical Applications of Laplace Transform in Differential Equations

The Laplace Transform is widely used in various engineering and physics fields, including:

- Control systems: Analyzing system stability and transient response.
- Electrical engineering: Solving circuit differential equations.
- Mechanical systems: Studying vibrations and damping phenomena.
- Signal processing: Analyzing system responses to inputs.

The ability to convert complex differential equations into manageable algebraic equations makes the Laplace Transform an invaluable tool in both theoretical and applied mathematics.

Conclusion

Applying the Laplace Transform to the differential equation $Y'' + 8Y' + 17Y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 2$ demonstrates a systematic approach to solving linear differential equations. By transforming to the s-domain, algebraically solving for $Y(s)$, and then performing the inverse transform, we obtained the explicit solution $y(t) = 2 e^{-4 t} \sin t$. This method underscores the power of the Laplace Transform in handling initial value problems efficiently and effectively.

Keywords:

Laplace Transform, Differential Equation, Second-Order Differential Equation, Initial Conditions, Inverse Laplace Transform, Algebraic Equation, Standard Lap

Frequently Asked Questions

How do you apply the Laplace transform to the differential equation Y'' + 8Y' + 17Y = 0 with initial conditions Y(0)=0 and Y'(0)=2?

You take the Laplace transform of each term in the differential equation, converting derivatives into algebraic expressions involving s and Y(s). Using the initial conditions, you substitute Y(0)=0 and Y'(0)=2 into the transformed equation to solve for Y(s).

What is the Laplace transform of the second derivative Y''(t) in the given differential equation?

The Laplace transform of Y''(t) is $s^2Y(s) - sY(0) - Y'(0)$. Given Y(0)=0 and Y'(0)=2, this becomes $s^2Y(s) - 2$.

How do initial conditions Y(0)=0 and Y'(0)=2 affect the Laplace transformed differential equation?

They allow substitution into the transformed equation, simplifying it by replacing Y(0) with 0 and Y'(0) with 2, which helps in solving for Y(s).

What is the general solution for Y(s) after applying the Laplace transform to the differential equation?

The transformed equation becomes $(s^2 + 8s + 17)Y(s) = 2$, leading to $Y(s) = 2 / (s^2 + 8s + 17)$.

How do you find the inverse Laplace transform to obtain y(t) from Y(s) in this problem?

You decompose Y(s) into partial fractions or recognize it as a standard form, then use inverse Laplace transform tables or methods to find y(t). In this case, Y(s) corresponds to a damped sinusoid.

What is the final explicit solution y(t) for the differential equation after applying the Laplace transform and inverse transform?

The solution is $y(t) = (2/\sqrt{(17)})\, e^{-4t} \sin(\sqrt{(17)}\, t)$, which satisfies the initial conditions and the differential equation.

Why is the Laplace transform method effective for solving this second-order differential equation with initial conditions?

Because it converts the differential equation into an algebraic equation in the s-domain, making it easier to incorporate initial conditions and solve for Y(s), then invert to find y(t).

Additional Resources

In-Depth Analysis of Solving the Differential Equation $Y''' + 8Y' + 17Y = 0$ with Initial Conditions Using the Laplace Transform

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Introduction

Differential equations are fundamental tools in mathematical modeling, capturing the dynamics of physical systems, engineering processes, and various scientific phenomena. Among these, linear ordinary differential equations with constant coefficients are particularly significant because they often admit explicit solutions. The Laplace Transform is a powerful method to solve such differential equations, especially when initial conditions are specified.

This review explores the comprehensive process of solving the third-order differential equation:

$$
Y''' + 8Y' + 17Y = 0,
$$

with initial conditions:

$$
Y(0) = 0, \quad Y'(0) = 2,
$$

using the Laplace Transform method. The analysis covers the theoretical foundation, step-by-step solution process, and interpretation of results, providing a deep understanding of the technique and its application.

Understanding the Differential Equation

General Form and Significance

The given differential equation is a linear, homogeneous, third-order differential equation with constant coefficients:

$$
Y''' + 8Y' + 17Y = 0.
$$

- Order: The highest derivative is third-order, indicating three initial conditions are typically needed for a unique solution.
- Linearity: The equation is linear, meaning the solution can be expressed as a superposition of solutions to the homogeneous equation.
- Constant Coefficients: Facilitates the use of algebraic methods, such as characteristic equations or Laplace Transforms.

Initial Conditions

- $Y(0) = 0$: The value of the solution at $t = 0$.
- $Y'(0) = 2$: The initial slope of the solution at $t = 0$.

Note that $Y''(0)$ is not specified, which suggests that the Laplace Transform will inherently incorporate the initial conditions for the derivatives as they appear.

The Laplace Transform: Theoretical Foundation

Definition

The Laplace Transform $\mathcal{L}\{f(t)\}$ of a function $f(t)$, defined for $t \geq 0$, is:

$$
F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st}f(t) \, dt,
$$

where s is a complex variable.

Key Properties

- Linearity: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$.
- Differentiation:

$$
\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).
$$

This property allows transforming derivatives into algebraic expressions in the s-

domain, incorporating initial conditions naturally.

Applying the Laplace Transform to the Differential Equation

Step 1: Take the Laplace Transform of Each Term

Transform each term individually:

1. Third derivative:
$$
\mathcal{L}\{Y'''\}\ = s^3Y(s) - s^2Y(0) - sY'(0) - Y''(0).
$$
Since $Y(0) = 0$ and $Y'(0) = 2$, this simplifies to:
$$
s^3Y(s) - 0 - s \times 2 - Y''(0) = s^3Y(s) - 2s - Y''(0).
$$
Note: $Y''(0)$ remains unknown at this point.

2. First derivative:
$$
\mathcal{L}\{Y'\} = sY(s) - Y(0) = sY(s) - 0 = sY(s).
$$

3. Function $Y(t)$:
$$
\mathcal{L}\{Y\} = Y(s).
$$

Putting it together, the transformed differential equation becomes:

$$
\left(s^3Y(s) - 2s - Y''(0) \right) + 8(sY(s)) + 17Y(s) = 0.
$$

Incorporating Initial Conditions and Unknowns

The transformed equation is:

$$
s^3Y(s) - 2s - Y''(0) + 8sY(s) + 17Y(s) = 0.
$$

Group the terms involving $Y(s)$:

$$
\left(s^3 + 8s + 17 \right) Y(s) = 2s + Y''(0).
$$

This leads to:

$$
Y(s) = \frac{2s + Y''(0)}{s^3 + 8s + 17}.
$$

The challenge here is that $Y''(0)$ is unknown. To proceed, we need to determine $Y''(0)$ using additional information, which is often provided by higher initial derivatives or boundary conditions. Since only $Y(0)$ and $Y'(0)$ are given, the approach involves solving the differential equation explicitly to find $Y''(0)$.

Solving the Differential Equation in the s-Domain

Step 1: Find the Characteristic Equation

Alternatively, since the differential equation is linear with constant coefficients, an easier approach—bypassing initial unknowns—is to find the characteristic roots directly:

$$
r^3 + 8r + 17 = 0.
$$

This is a cubic polynomial, and solving it reveals the nature of the solution (real roots or complex conjugates).

Step 2: Solve the Cubic Characteristic Equation

Attempting to factor or find roots:

- Rational Root Test suggests trying factors of 17 (±1, ±17).

Test $r = -1$:

$$
(-1)^3 + 8(-1) + 17 = -1 - 8 + 17 = 8 \neq 0.
$$

Test $r = 1$:

$$
1 + 8 + 17 = 26 \neq 0.
$$

Test $r = -17$:

$$
-4913 - 8(17) + 17 \neq 0.
$$

Test $r= 17$:

$$
4913 + 8(17) + 17 \neq 0.
$$

Thus, no rational roots. We need to use the cubic formula or numerical methods.

Numerical Solution of the Roots

Using the depressed cubic form or a calculator, the roots are approximately:

$$
r \approx -1.557 + 2.159i, \quad r \approx -1.557 - 2.159i, \quad r \approx 0.114.
$$

This indicates:

- Two complex conjugate roots with negative real parts.
- One real root slightly positive.

General Solution in the t-Domain

The general solution to the differential equation, based on roots, is:

$$
Y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + C_3 e^{r_3 t},
$$
where r_1, r_2, r_3 are the roots found above, and C_1, C_2, C_3 are constants determined by initial conditions.

Expressed explicitly:

$$
Y(t) = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t) + C_3 e^{\gamma t},
$$
where the complex conjugate roots correspond to sinusoidal terms multiplied by exponential decay, and the real root contributes an exponential term.

Applying Initial Conditions to Find Constants

Applying initial conditions directly in the time domain:

- $Y(0) = 0$,

- $Y'(0) = 2$,
- $Y''(0)$ can be found via differentiation or by substituting into the differential equation.

To find C_1, C_2, C_3, differentiate the general solution and evaluate at $t=0$:

$$
Y(0) = C_1 + C_2 + C_3 = 0.
$$

$$
Y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} + C_3 r_3 e^{r_3 t},
$$
and at $t=0$:

$$
Y'(0) = C_1 r_1 + C_2 r_2 + C_3 r_3 = 2.
$$

Similarly, $Y''(t)$ can be derived and evaluated at $t=0$ to involve the constants and roots, helping to determine $Y''(0)$.

Practical Approach: Using the Laplace Transform for a Complete Solution

Given the complexities in solving for $Y''(0)$ explicitly from initial conditions, the Laplace Transform method remains valuable

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