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Understanding the motion of particles in a plane is a fundamental aspect of classical mechanics, providing insights into oscillatory behavior, wave motion, and many real-world applications such as engineering, physics, and even biological systems. When the position of a particle in a plane is described by functions involving cosine or sine, it typically indicates oscillatory or periodic motion. In this article, we will explore the scenario where a particle's position is given by the functions $x(t) = A \cos \omega t$ and $y(t) = A \cos \omega t$, analyze the nature of its motion, derive key properties, and discuss various related concepts with detailed explanations.

Understanding the Position Functions and Their Significance

The Given Functions

The problem states that the position of a particle moving in a plane is described by:

$$
x(t) = A \cos \omega t
$$

$$
y(t) = A \cos \omega t
$$

Here:

- A is the amplitude, representing the maximum displacement from the origin.
- ω (omega) is the angular frequency, indicating how rapidly the particle oscillates.
- t is the time variable.

This form suggests that both the x and y coordinates oscillate cosinely with the same amplitude and frequency.

Implications of the Functions

- Since both coordinates depend on the same cosine function with identical amplitudes and frequencies, the particle's motion is highly symmetric.
- The particle's position at any time t is given by the point $(x(t), y(t))$ on the plane.
- The nature of the motion can be understood better by examining the relationship between $x(t)$ and $y(t)$.

Analyzing the Trajectory of the Particle

Deriving the Equation of the Path

Given:

$$
x(t) = A \cos \omega t
$$
$$
y(t) = A \cos \omega t
$$

Since both are equal, the particle's position traces a specific path. We can eliminate the time variable t:

$$
x(t) = y(t) \Rightarrow x = y
$$

or, more generally, if the functions are identical, the trajectory is along the line $y = x$. But in typical cases, the functions may differ in phase or amplitude. To consider the more general case, suppose:

$$
x(t) = A_x \cos (\omega t + \phi_x)
$$
$$
y(t) = A_y \cos (\omega t + \phi_y)
$$

In our initial case, with both equal and no phase difference:

$$
A_x = A_y = A, \quad \phi_x = \phi_y = 0
$$

then:

$$
x(t) = y(t) \Rightarrow \text{the particle moves along the line } y = x
$$

which is a straight line passing through the origin at a 45-degree angle.

If phase differences or different amplitudes are introduced, the trajectory becomes more complex, such as an ellipse or a circle.

Special case: Equal amplitude and same phase

- The particle moves along the line $y = x$.
- Since both coordinates are identical, the position vector is:

$$
\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} = A \cos \omega t (\hat{i} + \hat{j})
$$

- The magnitude of the position vector:

$$
r(t) = |\vec{r}(t)| = \sqrt{x(t)^2 + y(t)^2} = \sqrt{2} A |\cos \omega t|
$$

- The particle oscillates back and forth along the line $y = x$, with maximum displacement $\sqrt{2} A$.

Nature of the Motion: Oscillations and Trajectories

Types of Oscillatory Motion

Oscillations described by cosine functions are periodic and harmonic. The key features include:

1. **Period (T):** The time for one complete cycle of oscillation, given by:

   $$
   T = \frac{2\pi}{\omega}
   $$

2. **Frequency (f):** Number of oscillations per unit time:

   $$
   f = \frac{1}{T} = \frac{\omega}{2\pi}
   $$

3. **Amplitude (A):** Maximum displacement from equilibrium position.

In the specific case where both $x(t)$ and $y(t)$ are identical, the particle's motion is along a straight line with oscillations of amplitude A.

Circular and Elliptical Motion

When the functions involve phase differences or different amplitudes:

$$
x(t) = A_x \cos (\omega t)
$$
$$
y(t) = A_y \sin (\omega t)
$$

the trajectory becomes:

- Circle: if $A_x = A_y$ and phase difference is 90° (i.e., sine and cosine functions), the particle moves in a perfect circle.
- Ellipse: if $A_x \neq A_y$ with phase difference 90°, the motion traces an ellipse.
- Line: if phase difference is zero and amplitudes are equal, the motion is along a line.

In our initial case, with both functions identical, the motion is along the line $y = x$.

Velocity and Acceleration in the Motion

Calculating Velocity Components

The velocity components are the derivatives of position functions:

$$
v_x(t) = \frac{d x(t)}{dt} = -A \omega \sin \omega t
$$

$$
v_y(t) = \frac{d y(t)}{dt} = -A \omega \sin \omega t
$$

Since both are the same:

$$
\vec{v}(t) = v_x(t) \hat{i} + v_y(t) \hat{j} = -A \omega \sin \omega t (\hat{i} + \hat{j})
$$

This indicates the particle's velocity vector points along the same line as the position vector, with magnitude:

$$
v(t) = |\vec{v}(t)| = \sqrt{v_x^2 + v_y^2} = \sqrt{2} A \omega |\sin \omega t|
$$

Calculating Acceleration Components

Acceleration is the derivative of velocity:

$$
a_x(t) = \frac{d v_x(t)}{dt} = -A \omega^2 \cos \omega t
$$

$$
a_y(t) = \frac{d v_y(t)}{dt} = -A \omega^2 \cos \omega t
$$

- The acceleration vector:

$$
\vec{a}(t) = -A \omega^2 \cos \omega t (\hat{i} + \hat{j})
$$

- Magnitude of acceleration:

$$
a(t) = |\vec{a}(t)| = \sqrt{2} A \omega^2 |\cos \omega t|
$$

This confirms the harmonic nature of the particle's motion, with acceleration always directed toward the equilibrium position, characteristic of simple harmonic motion.

Energy Considerations in the Oscillatory Motion

Potential and Kinetic Energy

In harmonic oscillations, energy alternates between kinetic and potential forms:

- Kinetic Energy (KE):

$$
KE = \frac{1}{2} m v^2 = \frac{1}{2} m (2 A^2 \omega^2 \sin^2 \omega t) = m A^2 \omega^2 \sin^2 \omega t
$$

- Potential Energy (PE):

$$
PE = \frac{1}{2} k x^2
$$

where k is the effective spring constant related to the motion.

Given that the displacement is $x(t) = A \cos \omega t$, the maximum potential energy occurs at maximum displacement:

$$
PE_{max} = \frac{1}{2} k A^2
$$

Similarly, total mechanical energy remains constant:

$$
E_{total} = KE + PE = \text{constant}
$$

In our case, because both $x(t)$ and $y(t)$ oscillate with the same function, the total energy involves the combined motion along the line.

Applications and Real-World Examples

Oscillatory Systems in Engineering and Physics

The mathematical description of particle motion using cosine functions is prevalent in numerous physical systems:

1. **Simple Harmonic Oscillators:** Springs, pendulums, and LC circuits exhibit similar behavior.