In Each Part, Let S Be The Standard Basis For P2. Use The Results Proved In Exercises 22 And 23 To Find

In Each Part, Let S Be The Standard Basis For P2. Use The Results Proved In Exercises 22 And 23 To Find the various coordinate representations, change-of-basis matrices, and other related properties in polynomial vector spaces. This topic is central to understanding how different bases interact within vector spaces, especially when working with polynomial spaces such as P2, the space of all polynomials of degree at most 2. In this article, we will explore the theoretical foundations, practical methods, and applications of basis transformations, emphasizing the role of the standard basis as a reference point.

Understanding the Polynomial Space P2

Definition and Dimension

The space P2 consists of all polynomials of degree less than or equal to 2. Formally:

$$
P_2 = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}
$$

The dimension of P2 is 3, which means any basis of P2 must contain exactly three linearly independent polynomials.

Standard Basis for P2

The standard basis, often denoted as S, is:

$$
S = \{ 1, x, x^2 \}
$$

This basis is intuitive because each polynomial in P2 can be expressed uniquely as a linear combination of these three monomials.

Basis Transformations and Coordinate Vectors

Coordinate Vectors Relative to a Basis

Given a basis $B = \{ b_1, b_2, b_3 \}$ of P2, any polynomial $p \in P_2$ can be expressed as:

$$
p = c_1 b_1 + c_2 b_2 + c_3 b_3
$$

The coordinate vector of p relative to B, denoted $[p]_B$, is:

$$
[p]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
$$

When B is the standard basis S, $[p]_S$ directly contains the coefficients $a_0, a_1,$ and a_2.

Change of Basis Matrices

The transition from coordinates relative to one basis to another involves the change of basis matrix. If $B = \{b_1, b_2, b_3\}$ and S is the standard basis, the matrix P_B whose columns are the coordinate vectors of b_1, b_2, b_3 relative to S enables conversion:

$$
[p]_B = P_B^{-1} [p]_S
$$

Conversely:

$$
[p]_S = P_B [p]_B
$$

Understanding and computing these matrices is fundamental when shifting between bases, especially in polynomial interpolation, approximation, and solving linear systems.

Applying Results from Exercises 22 and 23

Exercises 22 and 23 typically involve computing the change of basis matrix and understanding how the basis vectors relate to the standard basis. Using these results, we can analyze several key problems:

Exercise 22: Finding the Change of Basis Matrix

Suppose the basis B for P2 is given as:

$$
b_1 = 2 + x, \quad b_2 = x^2 - 1, \quad b_3 = 3x + 4
$$

Express each basis vector in terms of the standard basis S:

$$
b_1 = 2 + x = 2 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}
$$

\[

$$
b_2 = -1 + 0 \cdot x + 1 \cdot x^2 \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
b_3 = 4 + 3x + 0 \cdot x^2 \Rightarrow \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}
$$

Construct the matrix P_B whose columns are these vectors:

$$
P_B = \begin{bmatrix}
2 & -1 & 4 \\
1 & 0 & 3 \\
0 & 1 & 0
\end{bmatrix}
$$

This matrix allows transforming coordinate vectors between bases.

Exercise 23: Computing Polynomial Coordinates

Given a polynomial $p(x) = 5x^2 + 2x + 1$, find its coordinate vector relative to basis B and then check the transformation to the standard basis:

$$
[p]_S = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}
$$

Compute:

$$
[p]_B = P_B^{-1} [p]_S
$$

This process highlights how the change of basis matrix simplifies the conversion and understanding of polynomial representations.

Practical Applications of Basis Transformations in P2

Polynomial Interpolation

Given a set of points, polynomial interpolation involves finding a polynomial that passes through all points. Choosing a basis affects the form and complexity of the interpolating polynomial. Using the standard basis simplifies coefficient extraction, but other bases (like Lagrange or Newton bases) may be more computationally efficient.

Approximation and Fourier Series

Transforming polynomials into different bases allows for better approximation techniques, such as Chebyshev or Legendre polynomial bases. The change-of-basis matrices facilitate these transformations, enabling approximation in optimal bases for specific problems.

Symbolic Computation and Computer Algebra Systems

Computer algebra systems rely heavily on basis transformations to manipulate polynomials efficiently. The ability to switch between bases programmatically is crucial for tasks like simplification, factorization, and solving polynomial equations.

Summary and Key Takeaways

- The standard basis $S = \{1, x, x^2\}$ of P2 provides a natural coordinate system for polynomials.
- Changing bases within P2 involves computing the change-of-basis matrix, which is constructed from the basis vectors expressed in the standard basis.
- Exercises 22 and 23 provide practical methods for constructing these matrices and applying them to polynomial coordinate transformations.
- Understanding these concepts enhances our ability to analyze polynomial spaces, perform interpolation, and manipulate polynomials in various applications.

Final Remarks

Mastering the use of the standard basis for P2 and the related change-of-basis techniques is fundamental for linear algebra, polynomial approximation, and computational mathematics. By carefully applying the results from exercises like 22 and 23, students and practitioners can efficiently transition between different polynomial representations, facilitating problem-solving and theoretical insights alike.

Whether working on theoretical proofs or practical computations, the concepts discussed here form a core part of the toolkit for anyone dealing with polynomial vector spaces and their transformations.

Frequently Asked Questions

What is the standard basis S for P2?

The standard basis S for P2 (the space of all polynomials of degree at most 2) is typically given as $S = \{1, x, x^2\}$.

How do the results from Exercises 22 and 23 assist in finding representations in P2?

Exercises 22 and 23 likely involve methods for expressing polynomials in terms of basis vectors, such as coordinate vectors or linear transformations, which help in representing polynomials relative to the standard basis S.

What is the process to find the coordinate vector of a polynomial in P2 relative to the standard basis S?

To find the coordinate vector, express the polynomial as a linear combination of the basis vectors 1, x, and x^2, and then write down the coefficients in order as a vector.

How can the results from previous exercises be used to find the transformation matrix from P2 to R^3?

By expressing each basis vector of P2 in coordinates and applying the results from Exercises 22 and 23, you can construct the matrix whose columns are the images of the basis vectors, representing the linear transformation.

What is the significance of using the standard basis S in polynomial vector spaces?

The standard basis provides a straightforward way to represent polynomials as coordinate vectors, simplifying calculations involving linear transformations, derivatives, and evaluations.

How do the results from Exercises 22 and 23 relate to linear transformations in P2?

They likely involve finding the matrix representations of linear transformations with respect to the basis S, enabling easier computation of images of polynomials under these transformations.

Can the methods from Exercises 22 and 23 be applied to find the image of a polynomial under a linear operator?

Yes, these methods help determine the coordinate vector of the polynomial and apply the transformation matrix to find its image.

What is the role of the standard basis in solving polynomial problems related to linear algebra?

The standard basis simplifies the process of expressing, transforming, and analyzing polynomials within linear algebra frameworks.

How do you verify that a set of polynomials forms a basis for P2 using the standard basis?

You check that the set spans P2 and is linearly independent; with the standard basis {1, x, x^2}, any polynomial in P2 can be uniquely expressed, confirming it is a basis.

What steps are involved in using the results from Exercises 22

and 23 to find a specific polynomial's image under a linear transformation?

First, express the polynomial as a coordinate vector relative to S, then multiply this vector by the transformation matrix obtained from the previous exercises to find the image's coordinate vector, and finally convert back if needed.

Additional Resources

In each part, let S be the standard basis for P_2. Use the results proved in Exercises 22 and 23 to find — this statement sets the stage for a detailed exploration into the structure and properties of polynomial vector spaces, specifically focusing on the space P_2 of all polynomials of degree at most 2. The phrase indicates a structured approach, where in each segment of our discussion, we will analyze different aspects of P_2 with respect to the standard basis, often denoted as $S = \{1, x, x^2\}$. By leveraging the results from earlier exercises, we can deepen our understanding of transformations, coordinate representations, and basis change operations within this finite-dimensional vector space.

Introduction to P_2 and the Standard Basis

What is P_2?

The vector space P_2 is the set of all polynomials with real coefficients of degree less than or equal to 2:

$$ P_2 = \left\{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \right\} $$

This space is three-dimensional because any polynomial in P_2 can be uniquely expressed as a linear combination of three basis elements.

The Standard Basis $S = \{1, x, x^2\}$

The standard basis for P_2, often denoted as S, consists of the polynomials:

$$ S = \{ 1, x, x^2 \} $$

This basis is canonical because each polynomial in P_2 can be represented as a coordinate vector relative to S:

$$ p(x) = a_0 \cdots + a_1 x + a_2 x^2 \quad \leftrightarrow \quad [p(x)]_S = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} $$

The standard basis simplifies many operations, including coordinate transformations and the analysis of linear maps.

Recap of Results from Exercises 22 and 23

Before diving into specific applications, let's briefly summarize the key results proved in Exercises 22 and 23, which will serve as tools for our subsequent analysis:

Exercise 22: Change of Basis and Coordinate Transformation

- Result: Given any basis $B = \{b_1, b_2, b_3\}$ of P_2, the coordinate vector of a polynomial p relative to B can be obtained via a change of basis matrix $P_{B \leftarrow S}$, which is constructed by expressing each basis vector b_i in terms of the standard basis.

- Implication: To convert coordinates from basis B to the standard basis, multiply by the change-of-basis matrix. Conversely, to go from standard coordinates to coordinates relative to B, multiply by the inverse of that matrix.

Exercise 23: Linear Transformations and Matrices

- Result: The matrix representation of a linear transformation $T: P_2 \to P_2$ with respect to the basis S can be computed by applying T to each basis vector, expressing the images in the basis S, and forming a matrix whose columns are these coordinate vectors.

- Implication: This process allows us to analyze linear operators in matrix form, making it easier to study properties like eigenvalues, eigenvectors, and diagonalization.

Part 1: Representing Polynomials and Coordinate Vectors

Expressing Polynomials in the Standard Basis

In P_2, any polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ can be represented as a coordinate vector relative to S:

$$[p(x)]_S = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

This simple correspondence allows easy computation of polynomial addition, scalar multiplication, and linear transformations.

Using Results from Exercises 22 and 23

Suppose we have another basis $B = \{b_1, b_2, b_3\}$, where:

- $b_1 = 1 + x$
- $b_2 = x^2$
- $b_3 = 1 - x + x^2$

To express a polynomial p in the basis B, we need the change of basis matrix $P_{B \leftarrow S}$:

1. Express basis vectors in standard coordinates:

- $b_1 = 1 + x \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $b_2 = x^2 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- $b_3 = 1 - x + x^2 \rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2. Form the change-of-basis matrix:

$$P_{B \leftarrow S} = \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{bmatrix}$$

3. Coordinate transformation:

- To find the B-coordinates of a polynomial p, multiply its standard coordinate vector by $P_{B \leftarrow S}^{-1}$.

This process demonstrates the utility of the results in Exercises 22 and 23 for converting between different bases.

Part 2: Linear Transformations of P_2

Defining a Linear Transformation $T: P_2 \to P_2$

Consider a linear transformation T defined by its action on the basis S:

- $T(1) = 2 + x$
- $T(x) = x^2$
- $T(x^2) = 3x + 4$

Using Exercise 23, to find the matrix representation of T:

1. Express T of each basis vector in standard coordinates:

- $T(1) = 2 + x \rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
- $T(x) = x^2 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- $T(x^2) = 3x + 4 \rightarrow \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

2. Construct the matrix $[T]_S$ with these vectors as columns:

$$[T]_S = \begin{bmatrix}
2 & 0 & 4 \\
1 & 0 & 3 \\
0 & 1 & 0
\end{bmatrix}$$

This matrix can now be used to compute the image of any polynomial p in P_2 by multiplying $[p]_S$ by $[T]_S$.

Application: Finding the Image of a Polynomial

Suppose $p(x) = 1 + 2x + 3x^2$, then:

$$[p]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Applying T:

$$[T(p)]_S = [T]_S \times [p]_S = \begin{bmatrix}
2 & 0 & 4 \\
1 & 0 & 3 \\
0 & 1 & 0
\end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix}
2(1) + 0(2) + 4(3) \\
1(1) + 0(2) + 3(3) \\
0(1) + 1(2) + 0(3)
\end{bmatrix} = \begin{bmatrix}
2 + 0 + 12 \\
1 + 0 + 9 \\
0 + 2 + 0
\end{bmatrix} = \begin{bmatrix}
14 \\ 10 \\ 2
\end{bmatrix}$$

The polynomial $T(p)$ in standard form is:

$$14 + 10x + 2x^2$$

Part 3: Change of Basis and Diagonalization

Transformations with Different Bases

Using the change-of-basis matrices from Exercise 22, we can analyze how linear transformations behave under different bases:

- Coordinate in basis B:

$$[p]_B = P_{B \leftarrow S}^{-1} [p]_S$$

- Transformation in basis B:

$$[T]_B = P_{B \leftarrow S}^{-1} [T]_S P_{B \leftarrow S}$$

This process allows us to find a basis in which the matrix of T is diagonal, simplifying the analysis of eigenvalues and eigenvectors.

Diagonalization in P_2

Suppose we want to diagonalize T:

1. Find eigenvalues by solving $\det([T]_S - \lambda I) =$

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