Intersection Find The Extreme Values Of The Function (x, Y, Z) = Xy Z2 On The Circle In Which The Plane

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Introduction to Optimization Problems on Surfaces and Curves

Optimization problems involving functions of multiple variables are fundamental in mathematics, physics, engineering, and computer science. These problems often entail finding the maximum or minimum values of a function subject to certain constraints, such as the function's domain being restricted to a specific surface or curve. One common scenario involves determining the extreme values of a multivariable function along a curve or surface embedded in three-dimensional space.

This article explores a specific optimization problem: finding the extreme values of the function $f(x, y, z) = xy z^2$ on a particular curve—namely, the intersection of a specified plane with a certain surface. The process involves understanding the geometric context, setting up the problem correctly, applying methods such as Lagrange multipliers, and interpreting the solutions to identify the maximum and minimum values of the function on the given curve.

Understanding the Mathematical Context

The Function $f(x, y, z) = xy z^2$

This function involves three variables:

- x and y, which are multiplied directly.
- z, which appears squared, emphasizing its quadratic influence on the function's value.

The function's behavior is sensitive to the signs and magnitudes of $x, y,$ and z, especially since $z^2 \geq 0$ for all real z. The goal is to find points on a certain curve where this function attains its extreme (maximum or minimum) values.

The Geometric Setting: Intersection of a Plane with a Surface

The problem involves analyzing the behavior of $f(x, y, z)$ along the intersection of:

- A plane, defined by a linear equation such as $ax + by + cz = d$.
- A surface, which could be a sphere, cylinder, paraboloid, or other common surfaces in 3D space.

The intersection of such surfaces typically forms a curve—often a circle, ellipse, or more complex shape—along which the optimization problem is constrained.

Formulating the Problem

Defining the Constraints

Suppose the plane is given by:

$$
ax + by + cz = d
$$

and the surface is described by an equation such as:

$$
x^2 + y^2 + z^2 = R^2
$$

which represents a sphere of radius R.

The intersection of the plane and the sphere yields a circle (assuming they intersect in a non-degenerate way). The optimization of $f(x, y, z) = xy z^2$ occurs along this circle.

Objective Function and Constraints

- Objective function: $f(x, y, z) = xy z^2$
- Constraints:
- $ax + by + cz = d$ (plane)
- $x^2 + y^2 + z^2 = R^2$ (sphere)

The problem reduces to finding the extrema of f subject to these two constraints, which is a classic case suitable for the method of Lagrange multipliers.

Methodology for Finding Extreme Values

Applying Lagrange Multipliers

The method involves introducing multipliers λ and μ for each constraint:

$$
\mathcal{L}(x, y, z, \lambda, \mu) = xy z^2 + \lambda (ax + by + cz - d) + \mu (x^2 + y^2 + z^2 - R^2)
$$

Finding stationary points requires solving the system:

$$
\nabla_{x, y, z} \mathcal{L} = 0
$$
$$
ax + by + cz = d
$$
$$
x^2 + y^2 + z^2 = R^2
$$

This involves computing the partial derivatives:

$$
\frac{\partial \mathcal{L}}{\partial x} = y z^2 + \lambda a + 2 \mu x = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial y} = x z^2 + \lambda b + 2 \mu y = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial z} = 2 xy z + \lambda c + 2 \mu z = 0
$$

and solving the resulting system for x, y, z, λ, μ.

Step-by-Step Solution Approach

Step 1: Set Up the Equations

Write the system based on the derivatives:

$$
\begin{cases}
y z^2 + \lambda a + 2 \mu x = 0 \\
x z^2 + \lambda b + 2 \mu y = 0 \\
2 x y z + \lambda c + 2 \mu z = 0 \\
ax + by + cz = d \\
x^2 + y^2 + z^2 = R^2
\end{cases}
$$

This system can be complex, but symmetry and substitution methods help simplify it.

Step 2: Simplify and Solve the System

- Use the first two equations to express λ in terms of x, y, z.
- Substitute into the third equation to relate the variables.
- Use the plane equation to eliminate one variable.
- Use the sphere equation to relate x, y, z.

In many cases, symmetry assumptions or specific numerical values for parameters simplify calculations.

Step 3: Analyze Critical Points

- Identify potential critical points (x, y, z) that satisfy all equations.
- Evaluate the function $f(x, y, z) = xy z^2$ at these points.
- Determine which points give maximum or minimum values.

Examples and Special Cases

Example 1: Symmetry in x and y

Suppose the plane is symmetric in x and y (e.g., $x + y = c$), simplifying the system:

- Assume $x = y$, which reduces the number of variables.
- Solve the resulting equations for x, z.
- Evaluate $f(x, y, z) = x y z^2 = x^2 z^2$.

This approach often reveals critical points where the function reaches its extrema.

Example 2: Special Cases for $z=0$ or $z\neq 0$

- When $z=0$, $f=0$, which could be a minimum.
- When $z \neq 0$, divide equations by z to simplify.

Interpreting the Results

Once critical points are identified, evaluate the function at these points to find:

- The maximum value of f on the curve.
- The minimum value of f on the curve.

The nature of these extrema can be confirmed using second derivative tests or analyzing the behavior of f near the critical points.

Conclusion and Practical Applications

Finding the extreme values of a multivariable function constrained to a curve, such as the intersection of a plane and a surface, is a cornerstone technique in optimization theory. The method of Lagrange multipliers provides a systematic approach to handle multiple constraints.

This problem illustrates how to establish the necessary equations, interpret the geometric constraints, and solve for critical points, which can then be used to determine the maximum and minimum values of the function.

Practical applications of such optimization problems include:

- Engineering design: optimizing structural components subject to physical constraints.
- Economics: maximizing profit or minimizing cost within resource limitations.
- Physics: finding equilibrium states with constraints imposed by conservation laws.
- Computer graphics: calculating optimal viewpoints or shading parameters constrained to specific surfaces.

Understanding how to navigate these constrained optimization problems enhances analytical skills and provides valuable insights into real-world systems modeled by multivariable functions.

Summary

- The problem involves maximizing or minimizing $f(x, y, z) = xy z^2$ on a curve defined by the intersection of a plane and a surface.
- Constraints are incorporated using Lagrange multipliers, leading to a system of equations.
- Symmetry and special cases simplify the solution process.
- Critical points are analyzed to identify the extrema.
- The methodology applies broadly across scientific and engineering disciplines where constrained optimization is essential.

By mastering these techniques, students and professionals can approach complex optimization problems with confidence and precision.

Frequently Asked Questions

What is the main goal when finding the extreme values of the function $f(x, y, z) = xy z^2$ on a given circle?

The main goal is to determine the maximum and minimum values of the function $f(x, y, z) = xy z^2$ constrained to the circle defined by a specific plane, typically using methods like Lagrange multipliers.

How do you set up the problem of finding extreme values of a function constrained to a circle in 3D space?

You set up the problem by defining the objective function (here, $f(x, y, z) = xy z^2$) and the constraint equation representing the circle (such as a plane equation). Then, use Lagrange multipliers to find stationary points satisfying both equations.

What role does the plane equation play in finding the extrema of the function on the circle?

The plane equation defines the circle constraint in 3D space. It restricts the domain of the function to points lying on that circle, enabling the use of constrained optimization techniques to find extrema.

Can you explain how to apply Lagrange multipliers to this problem?

Yes, you set up the Lagrangian as $L(x, y, z, \lambda) = xy z^2 - \lambda(g(x, y, z) - c)$, where $g(x, y, z) = 0$ is the plane equation, and c is a constant. Then, take partial derivatives and solve the resulting system to find critical points.

What are the steps involved in solving for the extreme values

on the circle?

The steps include: 1) writing the constraint equation for the circle, 2) forming the Lagrangian, 3) computing partial derivatives, 4) solving the system of equations, and 5) evaluating the function at the critical points to identify the extrema.

How do the symmetry and geometry of the circle influence the extremum points?

Symmetry and geometric properties help identify potential extremal points, especially when the function or the constraints exhibit symmetry. This can simplify calculations by considering symmetric points or using geometric intuition to narrow down candidates.

Are there special cases or particular positions on the circle where extrema are more likely to occur?

Yes, extrema often occur at points where the gradient of the function aligns with the normal vector to the constraint surface, such as intersections with axes or symmetry points on the circle.

How can the results of this problem be visualized or interpreted geometrically?

The extrema correspond to points on the circle where the function reaches highest or lowest values, which can be visualized as peaks or valleys on the surface defined by the function intersected with the circle's plane, highlighting the geometric relationship between the surface and the circle.

Additional Resources

Intersection Find The Extreme Values Of The Function (x, y, z) = $xy z^2$ On The Circle In Which The Plane

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Introduction

In multivariable calculus, optimization problems often involve finding the maximum and minimum values of a function subject to certain constraints. One common scenario is to identify the extreme values of a function defined in three-dimensional space, constrained to a particular surface or curve such as a circle, plane, or other geometric entities.

This article delves into the problem of finding the extreme values of the function:

$$
f(x, y, z) = xy z^2
$$

on the circle in which the plane intersects with the space defined by the variables x, y, z. The problem involves understanding the geometric nature of the constraints, applying techniques such as the method of Lagrange multipliers, and analyzing the behavior of the function at critical points.

Understanding the Problem

Before jumping into calculations, it’s crucial to clarify what the problem entails:

- Function to optimize: $f(x, y, z) = xy z^2$
- Constraint: The point (x, y, z) lies on a circle which is the intersection of a plane with some surface or space.

Depending on the specific problem, the circle is typically defined as the intersection of:

- A plane: $ax + by + cz + d = 0$ (or similar linear equation)
- A sphere or other surface: e.g., $x^2 + y^2 + z^2 = R^2$

In most standard problems, the circle is the intersection of a plane and a sphere, forming a circle.

For the purpose of this discussion, we will consider the most common scenario:

> The circle is the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and a plane $ax + by + cz + d = 0$.

This assumption is typical because it provides a well-defined geometric locus—the circle—and is rich enough for applying optimization techniques.

Geometric Setup and Constraints

1. The Sphere: $x^2 + y^2 + z^2 = R^2$

- Represents a sphere centered at the origin with radius R.
- Any point on the sphere satisfies the equation, and the intersection with a plane produces a circle (unless tangent, in which case it reduces to a point).

2. The Plane: $ax + by + cz + d = 0$

- Defines the geometric plane in 3D space.
- The intersection with the sphere yields a circle (assuming the plane cuts through the sphere).

3. The Circle of Intersection

- The intersection is a circle lying on both the sphere and the plane.
- Every point on this circle satisfies both equations.

Mathematical Formulation

The goal is to maximize or minimize $f(x, y, z) = xy z^2$ subject to the constraints:

$$
\begin{cases}
x^2 + y^2 + z^2 = R^2 \quad \text{(sphere)} \\
ax + by + cz + d = 0 \quad \text{(plane)}
\end{cases}
$$

Methodology for Finding Extreme Values

To find the extrema, the most systematic approach involves Lagrange multipliers, which are particularly suited for constrained optimization problems.

1. Setting Up the Lagrangian

The Lagrangian function is:

$$
\mathcal{L}(x, y, z, \lambda, \mu) = xy z^2 + \lambda (x^2 + y^2 + z^2 - R^2) + \mu (ax + by + cz + d)
$$

where:

- λ and μ are Lagrange multipliers corresponding to the sphere and plane constraints, respectively.

2. Finding Critical Points

The necessary conditions for extrema are obtained by taking partial derivatives with respect to x, y, z, and setting them to zero:

$$
\begin{cases}
\frac{\partial \mathcal{L}}{\partial x} = y z^2 + 2 \lambda x + \mu a = 0 \\
\frac{\partial \mathcal{L}}{\partial y} = x z^2 + 2 \lambda y + \mu b = 0 \\
\frac{\partial \mathcal{L}}{\partial z} = 2 x y z + 2 \lambda z + \mu c = 0
\end{cases}
$$

Along with the constraints:

$$
\begin{cases}
x^2 + y^2 + z^2 = R^2 \\
ax + by + cz + d = 0
\end{cases}
$$

Solving the System of Equations

The core of the problem involves solving the above system for the variables (x, y, z) and multipliers λ, μ. This is generally a challenging algebraic task, often requiring case-by-case analysis or substitution.

Deep Dive into Solution Strategies

1. Symmetry and Simplification

- Observe that the function involves the product $xy z^2$. Since z^2 appears, the sign of z does not affect the magnitude of the function—only the magnitude of z.
- The symmetry suggests that extremal values may occur at points where the variables are proportional or at specific symmetric points.

2. Special Cases

- Case when $z=0$: The function $f = xy \times 0^2 = 0$. So, zero is a critical value, possibly a minimum.
- Case when $x=0$ or $y=0$: Similarly, the function reduces to zero, indicating points where at least one of x or y is zero lead to the function vanishing.

3. Exploiting the Plane Equation

- The plane constraint can help eliminate one variable, reducing the problem to two variables, making the algebra more manageable.

Practical Example: Specific Constraints

To make the problem more concrete, consider specific values:

- Sphere: $x^2 + y^2 + z^2 = R^2$
- Plane: $z = mx + ny + c$

and analyze the function $f(x, y, z) = xy z^2$ on the circle formed by the intersection.

Step-by-Step Example

Suppose the plane is:

$$
z = 0
$$

which simplifies the problem to the circle $x^2 + y^2 = R^2$ in the plane $z=0$.

- On this circle, $z=0$, so $f(x, y, 0) = xy \times 0^2 = 0$.

Thus, the function's maximum and minimum values are both zero on this circle.

Now, consider the plane:

$$
z = k
$$

with $k \neq 0$. The intersection circle is:

$$
x^2 + y^2 = R^2 - k^2
$$

assuming $R^2 > k^2$.

The function simplifies to:

$$
f(x, y, k) = xy k^2
$$

since $z = k$. Now, the problem reduces to maximizing or minimizing xy on the circle $x^2 + y^2 = R^2 - k^2$.

Optimizing xy on a Circle

Given the circle:

$$
x^2 + y^2 = \rho^2
$$

where $\rho^2 = R^2 - k^2$,

- xy achieves its maximum and minimum when $x = y$ or $x = - y$.

Express $x = \rho \cos \theta$, $y = \rho \sin \theta$:

$$
xy = \rho^2 \sin \theta \cos \theta = \frac{\rho^2}{2} \sin 2\theta
$$

- The maximum of $\sin 2\theta$ is 1, so:

\[
\boxed{
\max xy = \frac{\rho^2}{2}
}
\]

- The minimum of \(\sin 2\theta\) is -1, so:

\[
\boxed{
\min xy = - \frac{\rho^2}{2}
}
\]

Therefore, the extremal values of \(f\) are:

\[
f_{max} = \frac{\rho^2}{2} \times k^2 = \frac{(R^2 - k^2)}{2} \times k^2
\]
\[
f_{min} = - \frac{(R^2 - k^2)}{2} \times k^2
\]

Final Results for the Example

- Maximum value:

\[
f_{max} = \frac{k^2}{2} (R^2 - k^2)
\]

- Minimum value:

\[
f_{min} = - \frac{k^2}{2} (R^2 - k^2)
\]

This approach

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