Let F Be The Function Given By F(x)= X^3 6x^2 15x. What Is The Maximum Value Of F On The Interval [0,6]

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Understanding how to find the maximum value of a function within a specific interval is a fundamental concept in calculus. In this article, we will analyze the function F(x) = x^3 + 6x^2 + 15x, and determine its maximum value over the interval [0,6]. We will explore the necessary calculus techniques, including derivatives, critical points, and the evaluation of endpoints, to comprehensively solve this problem. This guide aims to provide clear, step-by-step instructions suitable for students and enthusiasts seeking to deepen their understanding of optimization problems.

Understanding the Function F(x) = x^3 + 6x^2 + 15x

Before diving into the calculation, it’s important to understand the behavior of the function:

The structure of F(x)

- The function combines a cubic term (x^3), a quadratic term (6x^2), and a linear term (15x).
- The coefficients suggest that as x increases, the cubic term will dominate for large x, influencing the overall shape.
- Since the interval is [0,6], analyzing the function’s behavior within this domain is essential to find its maximum.

Graphical intuition

- For small x, the function starts near 0.
- The cubic nature indicates potential points of inflection and local maxima or minima.
- The function’s increasing or decreasing trend within [0,6] will determine the maximum.

Step 1: Express the Function Clearly

For clarity, rewrite the function:

$$ F(x) = x^3 + 6x^2 + 15x $$

Our goal is to find:

$$ \max_{x \in [0,6]} F(x) $$

Step 2: Find the Critical Points via Derivative

Critical points occur where the derivative equals zero or is undefined. Since F(x) is a polynomial, its derivative exists everywhere, so we focus on setting the derivative to zero:

$$ F'(x) = \frac{d}{dx}(x^3 + 6x^2 + 15x) $$

Calculating the derivative:

$$ F'(x) = 3x^2 + 12x + 15 $$

Step 3: Solve for Critical Points

Set the derivative equal to zero:

$$ 3x^2 + 12x + 15 = 0 $$

Divide through by 3 to simplify:

$$ x^2 + 4x + 5 = 0 $$

Use the quadratic formula:

$$ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1$, $b=4$, $c=5$:

$$ x = \frac{-4 \pm \sqrt{16 - 20}}{2} $$
$$ x = \frac{-4 \pm \sqrt{-4}}{2} $$

Since the discriminant $\sqrt{-4}$ is imaginary, there are no real critical points.

Implication: The derivative never equals zero on the real line, meaning F(x) has no local maxima or minima within the interval. The function is strictly increasing or decreasing.

Step 4: Determine the Monotonicity of F(x)

Because the derivative is:

$$ F'(x) = 3x^2 + 12x + 15 $$

and the discriminant is negative, the quadratic has no real roots, and since $a=3 > 0$, the parabola opens upward, indicating:

$$ F'(x) > 0 \quad \text{for all real } x $$

Conclusion: F(x) is strictly increasing over its entire domain, including [0,6].

Step 5: Evaluate the Function at Endpoints

Since F(x) is strictly increasing, the maximum value on [0,6] occurs at the right endpoint $x=6$.

Calculate:

$$ F(0) = 0^3 + 6 \times 0^2 + 15 \times 0 = 0 $$

$$ F(6) = 6^3 + 6 \times 6^2 + 15 \times 6 $$

Compute each term:

- $6^3 = 216$
- $6 \times 6^2 = 6 \times 36 = 216$
- $15 \times 6 = 90$

Sum:

$$ F(6) = 216 + 216 + 90 = 522 $$

Step 6: Final Result — Maximum Value of F on [0,6]

Given that the function is strictly increasing:

- The maximum value occurs at $x=6$.
- The maximum value is $\boxed{522}$.

Summary of Findings

- The derivative $F'(x) = 3x^2 + 12x + 15$ is always positive.
- No critical points exist within the interval, so the function is strictly increasing.
- The maximum value over [0,6] is at the endpoint $x=6$.
- The maximum value of F on [0,6] is 522.

Additional Insights and Applications

Understanding how to analyze polynomial functions over specific intervals has many practical applications:

1. **Optimization Problems:** Finding maximum profit, efficiency, or output in real-world scenarios.

2. **Physics:** Determining maximum displacement or velocity in motion problems.

3. **Economics:** Maximizing revenue or minimizing costs within constraints.

The process outlined above—derivative calculation, critical point analysis, and endpoint evaluation—is fundamental in calculus and widely applicable across disciplines.

Conclusion

In conclusion, for the function $F(x) = x^3 + 6x^2 + 15x$ over the interval [0,6], the maximum value is attained at the endpoint $x=6$, with a value of 522. The key steps involved were calculating the derivative, analyzing its sign to determine the monotonicity, and evaluating the function at the interval's endpoints. This approach exemplifies the power of calculus in solving optimization problems and provides a solid foundation for tackling more complex functions and constraints.

FAQs

1. Why is there no critical point inside the interval?

Because the derivative $F'(x) = 3x^2 + 12x + 15$ has no real roots (discriminant negative), indicating the function has no stationary points where the slope is zero. The function is thus strictly increasing or decreasing over the entire domain.

2. Can the maximum value occur at a critical point?

Yes, but only if such a point exists within the interval. Since no critical points exist for $F(x)$ in [0,6], the maximum must be at the endpoints.

3. How do I find the minimum value of F(x) on [0,6]?

Given the strictly increasing nature of F(x), the minimum occurs at the left endpoint $x=0$, with $F(0) = 0$.

4. How does this process help in real-world applications?

It allows professionals to find optimal points—like maximum profit or minimum cost—by analyzing the behavior of functions representing real-world quantities.

In summary, understanding the behavior of polynomial functions through derivatives and endpoint evaluation is essential for solving maximum and minimum problems efficiently. The problem of finding the maximum value of $F(x) = x^3 + 6x^2 + 15x$ over [0,6] demonstrates the straightforward application of calculus principles, culminating in the conclusion that the maximum value is 522 at $x=6$.

Frequently Asked Questions

What is the given function F(x) in the problem?

$F(x) = x^3 - 6x^2 + 15x$

What is the interval on which we need to find the maximum value of F?

The interval is [0, 6].

How do we find the maximum value of F on [0, 6]?

By evaluating the critical points inside the interval and the endpoints, then comparing their F(x) values.

What are the steps to find the critical points of F(x)?

Calculate the derivative F'(x), set it equal to zero, and solve for x.

What is the derivative F'(x) of the function?

$F'(x) = 3x^2 - 12x + 15$.

What are the critical points of F(x) within [0, 6]?

Solve $3x^2 - 12x + 15 = 0$, which simplifies to $x^2 - 4x + 5 = 0$. The solutions are complex, so there are no critical points inside the interval.

What is the maximum value of F(x) on [0, 6], given the critical points and endpoints?

Evaluate F at x=0 and x=6: $F(0)=0$, $F(6)=6^3 - 66^2 + 156 = 216 - 216 + 90 = 90$. Since no critical points are in [0,6], the maximum is at x=6 with $F(6)=90$.

Additional Resources

Optimization Analysis of the Function $F(x) = x^3 - 6x^2 + 15x$ on the Interval [0, 6]

Introduction

In the realm of mathematical analysis, particularly in calculus, understanding the behavior of functions within specified intervals is of paramount importance. Whether for academic pursuits, engineering applications, or data modeling, identifying maximum and minimum values—collectively known as extrema—forms a foundational skill. Today, we delve into a comprehensive examination of the function:

$$F(x) = x^3 - 6x^2 + 15x$$

defined over the closed interval $[0, 6]$. Our goal? To determine the maximum value of $F(x)$ within this interval, providing a meticulous, step-by-step analysis that combines theoretical rigor with practical insight.

Understanding the Function

Before diving into calculus tools, it's helpful to grasp the nature of $F(x)$. This cubic polynomial combines terms of varying degrees, suggesting that its graph exhibits a combination of increasing and decreasing behaviors, possibly with local maxima and minima.

Key characteristics:

- Degree: 3 (cubic), which typically results in a graph that can have one or two turning points.
- Leading coefficient: 1 (positive), indicating that as x approaches infinity, $F(x)$ tends toward infinity.
- Intercepts: The roots of $F(x)$ occur where $F(x) = 0$, which we will examine further.

Step 1: Critical Points and Derivative Analysis

To locate the extrema of $F(x)$, we employ calculus—specifically, the first derivative test. The critical points occur where the first derivative $F'(x)$ is zero or undefined. Since $F(x)$ is a polynomial, its derivative exists everywhere, simplifying our task.

Compute the derivative:

$$
F'(x) = \frac{d}{dx} (x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15
$$

Step 2: Find Critical Points

Set the derivative equal to zero:

$$
3x^2 - 12x + 15 = 0
$$

Divide through by 3 for simplicity:

$$
x^2 - 4x + 5 = 0
$$

Apply the quadratic formula:

$$
x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2} = \frac{4 \pm \sqrt{16 - 20}}{2}
$$

Simplify under the square root:

$$
x = \frac{4 \pm \sqrt{-4}}{2}
$$

Since the discriminant is negative (-4), there are no real solutions. This indicates that:

No critical points exist within the real number system.

Implication:

Because the derivative doesn't cross zero, the function has no internal extrema; it is either strictly increasing or decreasing over the interval. However, to confirm this, examine the sign of $F'(x)$ over $[0, 6]$.

Step 3: Sign of the Derivative $F'(x)$

Select a test point within the interval, say $x=0$:

$$
F'(0) = 3(0)^2 - 12(0) + 15 = 15 > 0
$$

Now, pick $x=3$:

$$
F'(3) = 3(9) - 12(3) + 15 = 27 - 36 + 15 = 6 > 0
$$

And $x=6$:

$$
F'(6) = 3(36) - 12(6) + 15 = 108 - 72 + 15 = 51 > 0
$$

Since $F'(x)$ is positive at multiple points, and the quadratic derivative factors are continuous, the derivative is positive throughout $[0, 6]$.

Conclusion:

The function $F(x)$ is strictly increasing on $[0, 6]$.

Step 4: Evaluate Endpoints

Given that $F(x)$ is strictly increasing and no internal maxima or minima exist within the interval, the maximum and minimum values must occur at the endpoints.

- At $x=0$:

$$
F(0) = 0^3 - 6 \times 0^2 + 15 \times 0 = 0
$$

- At $x=6$:

$$
F(6) = 6^3 - 6 \times 6^2 + 15 \times 6 = 216 - 6 \times 36 + 90 = 216 - 216 + 90 = 90
$$

Final Result:

- Minimum value of $F(x)$ on $[0,6]$: $F(0) = 0$
- Maximum value of $F(x)$ on $[0,6]$: $F(6) = 90$

Therefore, the maximum value of F on $[0, 6]$ is $\boxed{90}$, attained at $x=6$.

Additional Insights: Behavior of the Function

While our analysis confirms the monotonically increasing nature of $F(x)$ on $[0, 6]$, it's worthwhile to consider the broader behavior:

- Graphical intuition: The graph starts at $(0,0)$ and steadily increases to $(6, 90)$.
- No internal extrema: The absence of critical points within the interval simplifies the optimization problem.
- Application perspective: Such monotonic functions are straightforward in optimization contexts—they achieve their maximum at the right endpoint and their minimum at the left.

Practical Implications and Recommendations

Understanding the extremal values of functions like $F(x)$ is crucial for various real-world applications, including:

- Engineering: Determining maximum load, stress, or efficiency within specified parameters.
- Economics: Identifying optimal points such as maximum profit or minimum cost within constraints.
- Data Modeling: Recognizing the bounds of a model's output over a given domain.

In this particular case, knowing that $F(x)$ is increasing on $[0, 6]$ can guide decision-making processes, ensuring that evaluations are performed at the interval endpoints.

Summary

- The derivative $F'(x) = 3x^2 - 12x + 15$ has no real roots, indicating no internal critical points.
- The derivative is positive throughout $[0, 6]$, confirming that $F(x)$ is strictly increasing.
- The minimum value occurs at the start: $F(0) = 0$.
- The maximum value occurs at the end: $F(6) = 90$.

Hence, the maximum value of F on $[0, 6]$ is $\boxed{90}$, attained at $x=6$.

This comprehensive analysis exemplifies how calculus tools can effectively determine extrema, simplifying seemingly complex problems into straightforward evaluations.

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