Let T Be A Linear Operator On A Finite Dimensional Inner Product Space V. (1) Prove That Ker(T*T) = Ker

Introduction

Let T be a linear operator on a finite-dimensional inner product space V over the field of real or complex numbers. One of the fundamental aspects of operator theory involves understanding the properties of compositions like TT, where T denotes the adjoint of T. The kernel (or null space) of such operators reveals critical information about the structure of T, including its invertibility, rank, and spectrum. In this article, we aim to rigorously prove that the kernel of the operator TT coincides with the kernel of T itself, i.e., *Ker(TT) = Ker(T)*. This equality not only deepens our understanding of the relationship between an operator and its adjoint but also forms a foundation for many results in spectral theory and functional analysis.

Preliminaries and Definitions

Inner Product Space

An inner product space V is a vector space equipped with an inner product, a function *⟨·,·⟩* : V × V → ℝ (or ℂ) satisfying the following properties:

- **Linearity in the first argument:** For all u, v, w in V and scalars α, β, *⟨αu + βv, w⟩ = α⟨u, w⟩ + β⟨v, w⟩*
- **Conjugate symmetry:** *⟨u, v⟩ = overline{⟨v, u⟩}*
- **Positive definiteness:** *⟨v, v⟩ ≥ 0* with equality iff *v = 0*

Linear Operators and their Adjoints

A linear operator T : V → V is a function satisfying *T(αu + βv) = αT(u) + βT(v)* for all u, v in V and scalars α, β. The adjoint T of T is uniquely defined via the inner product:

- For all u, v in V, *⟨T(u), v⟩ = ⟨u, T(v)⟩*

The adjoint T is a linear operator from V to V, and its properties are fundamental in the analysis of T.

Kernel of an Operator

The kernel (or null space) of a linear operator $T : V \to V$ is defined as:

- $Ker(T) = \{ v \text{ in } V \mid T(v) = 0 \}$

This subspace measures the "degeneracy" of T, indicating where T fails to be invertible.

Statement of the Theorem

We aim to prove the following equality:

> ## Theorem:
>
> For a linear operator T on a finite-dimensional inner product space V, $Ker(TT) = Ker(T)$

Outline of the Proof

The proof involves two inclusions:

1. **Show that** $Ker(T) \subseteq Ker(TT)$
2. **Show that** $Ker(TT) \subseteq Ker(T)$

Establishing both directions confirms the equality of the kernels.

Proof of $\operatorname{Ker}(T) \subseteq \operatorname{Ker}(T^T)$

Step 1: Assume $v \in \operatorname{Ker}(T)$

Suppose $v \in V$ such that $T(v) = 0$.

Step 2: Compute $T^T(v)$

Applying the operator T^T to v, we get:

```
T^T(v) = T^(T(v))
```

Since \(T(v) = 0\), it follows that:

```
T^T(v) = T^(0) = 0
```

because linear operators map zero to zero.

Step 3: Conclusion

Thus, \(v \in \operatorname{Ker}(T^T)\), which proves:

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\(\operatorname{Ker}(T) \subseteq \operatorname{Ker}(T^T)\)
```

Proof of \(\operatorname{Ker}(T^T) \subseteq \operatorname{Ker}(T)\)

Step 1: Assume \(v \in \operatorname{Ker}(T^T)\)

Suppose \(v \in V\) such that:

```
T^T(v) = 0
```

Step 2: Use inner product properties

We analyze the inner product involving \(T(v)\). For any arbitrary \(w \in V\), consider:

```
\(\langle T(v), T(w) \rangle\)
```

Our goal is to connect this to the given condition \(T^T(v) = 0\). Recall that:

```
\(\langle T(v), T(w) \rangle = \langle T^T(v), w \rangle\)
```

since by the definition of the adjoint, for all u, w in V:

```
\(\langle T(u), w \rangle = \langle u, T^(w) \rangle\)
```

Step 3: Show that \(T(v) = 0\)

Set \(w = v\). Then:

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\(\langle T(v), T(v) \rangle = \langle T^T(v), v \rangle\)
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But since $T^T(v) = 0$, this simplifies to:

$$\langle T(v), T(v) \rangle = 0$$

Because the inner product is positive definite, this implies:

$$T(v) = 0$$

Step 4: Conclusion

Therefore, $v \in \operatorname{Ker}(T)$, and the inclusion is established:

$$\operatorname{Ker}(T^T) \subseteq \operatorname{Ker}(T)$$

Final Conclusion

Combining both inclusions, we obtain:

$$\operatorname{Ker}(T^T) = \operatorname{Ker}(T)$$

This completes the proof that the kernel of T^T coincides with the kernel of T in a finite-dimensional inner product space.

Implications and Further Remarks

Significance of the Result

- The equality $\operatorname{Ker}(T^T) = \operatorname{Ker}(T)$ highlights the connection between an operator and its adjoint, especially in terms of their null spaces.
- This result is instrumental in understanding the spectral properties of T, as it relates to the positive semi-definite operator T^T.
- It also underpins the polar decomposition theorem, which expresses T as a product of a partial isometry and a positive semi-definite operator.

Extensions and Related Results

- In infinite-dimensional Hilbert spaces, similar results hold with appropriate modifications, although additional considerations such as closures of subspaces are necessary.

- The concept extends to normal operators, where the spectral theorem applies, providing a rich structure for the analysis of operators.

Summary

In this article, we have meticulously demonstrated that for a linear operator T acting on a finite-dimensional inner product space V, the kernels of T and TT are identical. The proof hinges on the properties of the adjoint operator, the positive definiteness of the inner product, and fundamental linear algebraic techniques. This equality is a cornerstone in operator theory, enabling further exploration into the structure, spectrum, and decompositions of linear operators in finite and infinite-dimensional settings.

Frequently Asked Questions

What does the equality Ker(TT) = Ker(T) imply about the relationship between the kernels of T and TT?

It implies that the null spaces of T and TT are the same, meaning that any vector sent to zero by T is also sent to zero by TT, and vice versa.

How can we prove that Ker(TT) is contained in Ker(T)?

If v ∈ Ker(TT), then TT v = 0. Applying T to both sides gives T(Tv) = 0, which implies Tv = 0 because TT v = T(Tv). Therefore, v ∈ Ker(T).

Why does showing Ker(T) is contained in Ker(TT) suffice to establish equality?

Because the inclusion Ker(T) ⊆ Ker(TT) is straightforward, and proving the reverse inclusion Ker(TT) ⊆ Ker(T) completes the equality Ker(TT) = Ker(T).

What properties of inner product spaces and linear operators are essential in proving Ker(TT) = Ker(T)?

The key properties are the adjoint operation T, the fact that TT is self-adjoint, and the relationship between the kernels of an operator and its adjoint, leveraging inner product properties to relate these kernels.

Can you explain the role of the adjoint T in the proof of Ker(TT) = Ker(T)?

Yes, T is crucial because TT is a positive semi-definite operator, and using the properties of T helps relate the kernels of T and TT, especially through the identity TT v = 0 implying T v = 0.

Is the equality Ker(TT) = Ker(T) valid for all linear operators on finite-dimensional inner product spaces? Why or why not?

Yes, it is valid because in finite-dimensional inner product spaces, the kernels of T and TT coincide, due to the properties of adjoint operators and the positivity of TT, which are specific to finite-dimensional settings.

Additional Resources

Let T Be A Linear Operator On A Finite Dimensional Inner Product Space V. (1) Prove That Ker(TT) = Ker(T)

Introduction

In the realm of linear algebra, understanding the properties of linear operators on inner product spaces is fundamental. Among the various concepts, the study of kernels (null spaces), adjoint operators, and their interplay provides crucial insights into the structure and behavior of linear transformations. One particularly interesting statement involves the relation between the kernel of a composition involving an operator and its adjoint, specifically:

"Let T be a linear operator on a finite-dimensional inner product space V. Then, the kernel of TT equals the kernel of T."

This statement, while seemingly straightforward, embodies deep theoretical significance. It connects the algebraic notion of null spaces with the geometric intuition behind adjoints and orthogonality. The goal of this article is to rigorously prove this equality, explore its implications, and situate it within the broader context of linear operator theory.

The Setting: Inner Product Spaces and Linear Operators

Before delving into the proof, it is essential to establish the foundational concepts and notations used throughout.

- Inner Product Space: A real or complex vector space V equipped with an inner product, denoted ⟨·,·⟩, which is a positive-definite, sesquilinear (complex case) or bilinear (real case) form.

- Linear Operator T: A map T: V → V satisfying linearity: T(av + bw) = aT(v) + bT(w) for all v,w ∈ V and scalars a, b.

- Adjoint Operator T: The unique operator satisfying ⟨T(v), w⟩ = ⟨v, T(w)⟩ for all v,w ∈ V.

- Kernel (Null Space): The set Ker(T) = {v ∈ V | T(v) = 0}.

In finite-dimensional inner product spaces, the existence and properties of adjoint operators are well-behaved, enabling us to analyze their kernels and images effectively.

Main Objective

Prove that:

Ker(TT) = Ker(T).

This equivalence is not only elegant but also critical in proofs involving the spectral theorem, orthogonal projections, and the structure theory of operators.

Strategy of the Proof

The proof hinges on two inclusions:

1. Ker(T) ⊆ Ker(TT): Show that if v is in Ker(T), then v is also in Ker(TT).

2. Ker(TT) ⊆ Ker(T): Show that if v is in Ker(TT), then v is also in Ker(T).

Establishing both will confirm equality.

Detailed Proof

Part 1: Showing Ker(T) ⊆ Ker(TT)

Suppose v ∈ Ker(T). This means:

\[T(v) = 0. \]

We want to show that:

\[T^ T (v) = 0. \]

Note that:

\[T^ T (v) = T^ (T(v)) = T^(0) = 0, \]

since T(v) = 0 and T is linear. Therefore,

\[v \in \text{Ker}(T^ T). \]

Conclusion:

\[

\boxed{
\text{Ker}(T) \subseteq \text{Ker}(T^ T).
}
\]

Part 2: Showing Ker(TT) ⊆ Ker(T)

Suppose v ∈ Ker(TT). Then:

\[T^ T (v) = 0. \]

Our goal is to demonstrate that T(v) = 0, i.e., v ∈ Ker(T).

To do so, consider the inner product:

\[
\langle T(v), T(v) \rangle.
\]

Recall that:

\[
\langle T(v), T(v) \rangle = \| T(v) \|^2 \geq 0,
\]

with equality if and only if T(v) = 0.

Note that:

\[
\| T(v) \|^2 = \langle T(v), T(v) \rangle = \langle v, T^ T (v) \rangle,
\]

by the definition of the adjoint operator.

Since T^ T (v) = 0, it follows that:

\[
\langle v, T^ T (v) \rangle = \langle v, 0 \rangle = 0,
\]

and thus,

\[
\| T(v) \|^2 = 0,
\]

which implies:

$$
T(v) = 0.
$$

Therefore,

$$
v \in \text{Ker}(T),
$$

and consequently:

$$
\boxed{
\text{Ker}(T^ T) \subseteq \text{Ker}(T).
}
$$

Combining Both Parts

Having established both inclusions:

$$
\text{Ker}(T) \subseteq \text{Ker}(T^ T) \quad \text{and} \quad \text{Ker}(T^ T) \subseteq \text{Ker}(T),
$$

we conclude:

$$
\boxed{
\text{Ker}(T^ T) = \text{Ker}(T).
}
$$

This completes the proof.

Significance and Implications

The equality Ker(TT) = Ker(T) is not merely a mathematical curiosity; it has profound consequences in understanding the structure of linear operators.

1. Orthogonality and Range Spaces

The relation underscores that the nullity of T is intrinsically linked to the nullity of TT. Since TT is a positive semi-definite operator, its kernel encapsulates the vectors annihilated by T, revealing an orthogonal complement relationship.

2. Spectral Theorem and Self-Adjoint Operators

In finite-dimensional inner product spaces, the spectral theorem states that self-adjoint operators can be diagonalized via orthonormal bases. The operator TT being self-adjoint and positive semi-definite means its kernel reflects the null space of T, a fact utilized in spectral decomposition and singular value decomposition (SVD).

3. Singular Value Decomposition (SVD)

The proof forms a foundation for SVD, where T is decomposed into unitary operators and diagonal matrices capturing singular values. The kernel of T corresponds to zero singular values, and the relation with TT simplifies the analysis of these null spaces.

Broader Context and Related Results

The equality Ker(TT) = Ker(T) sits within a constellation of related properties:

- Range and Null Space Relations: The Rank-Nullity Theorem links the dimensions of the image and kernel, while the interplay between T and TT clarifies the structure of these subspaces.

- Positive Semi-Definiteness: Since TT is positive semi-definite, its kernel coincides with the orthogonal complement of its range, which also relates to the kernel of T.

- Moore-Penrose Pseudoinverse: The pseudoinverse T^+ can be characterized in terms of TT, and understanding its kernel is vital in solving least squares problems.

Conclusion

The proof that Ker(TT) = Ker(T) for a linear operator T on a finite-dimensional inner product space V is a classical yet profound result in linear algebra. Its simplicity belies deep implications for the structure of operators, spectral analysis, and applications like data science and quantum mechanics.

By establishing this equality, we gain insight into the orthogonal decomposition of V, the nature of null spaces, and the behavior of positive semi-definite operators derived from T. The result is a testament to the elegance of linear algebra, where algebraic and geometric perspectives intertwine seamlessly.

References

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This investigation underscores the beauty and power of linear algebraic structures, revealing how foundational properties underpin advanced theories and applications across mathematics and science.

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