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Introduction

Understanding the properties of points randomly selected within geometric shapes is a fundamental aspect of probability theory and geometry. When dealing with a circle of radius 1, centered at a specific point (often the origin), the question of how points are distributed within the circle becomes particularly interesting, especially when the points are chosen uniformly.

This article explores the probability distribution of the coordinates (X, Y) of a point uniformly selected within a circle of radius 1 centered at the origin. We will delve into the geometric setup, derive the probability density functions, analyze their properties, and discuss relevant applications in various fields such as physics, computer science, and engineering.

Geometric Setup and Basic Definitions

The Circle of Radius 1

Consider a circle centered at the origin $O(0,0)$ with radius 1. The equation of this circle is:

$$
x^2 + y^2 \leq 1
$$

Any point (X, Y) inside this circle satisfies this inequality.

Uniform Distribution in the Circle

When a point is chosen uniformly within the circle, it means that every region within the circle has an equal probability density per unit area. The total area of the circle is:

$$
A = \pi \times 1^2 = \pi
$$

Thus, the probability density must be uniform over this area, leading to specific properties for the distributions of X and Y.

Distribution of Coordinates (X, Y)

Symmetry and Independence

Because of the symmetry of the circle about both axes, the distributions of X and Y are identical. Furthermore, the joint distribution of (X, Y) is uniform over the disk, but the marginal distributions of X and Y are not uniform over the interval $[-1, 1]$.

Deriving the Probability Density Function (PDF)

Given that (X, Y) are uniformly distributed over the disk, the joint probability density function (PDF) is:

$$
f_{X,Y}(x, y) = \frac{1}{\pi}, \quad \text{for} \quad x^2 + y^2 \leq 1
$$

and zero elsewhere.

This uniform density over the area implies that the probability of the point falling within any subregion $R \subseteq$ circle is proportional to the area of R.

Marginal Distributions of X and Y

To find the marginal distribution of X, integrate the joint PDF over y:

$$
f_X(x) = \int_{-\sqrt{1 - x^2}}^{\sqrt{1 - x^2}} f_{X,Y}(x, y) \, dy
$$

Substituting $f_{X,Y}(x, y) = \frac{1}{\pi}$:

$$
f_X(x) = \frac{1}{\pi} \times 2 \sqrt{1 - x^2} = \frac{2}{\pi} \sqrt{1 - x^2}
$$

for $x \in [-1, 1]$.

Similarly, the distribution of Y is identical:

$$
f_Y(y) = \frac{2}{\pi} \sqrt{1 - y^2}
$$

for $y \in [-1, 1]$.

Visualizing the Distributions

The marginal PDF $f_X(x)$ (and similarly $f_Y(y)$) is a semi-elliptical shape, peaking at 0 and tapering to zero at ± 1. This shape is characteristic of the projection of the uniform distribution over the disk onto the axes.

Properties of the Distribution

Expected Values and Variance

Given the symmetry, both X and Y have:

$$
E[X] = E[Y] = 0
$$

due to symmetry about the origin.

The variance of X (and similarly Y) can be computed as:

$$
Var(X) = E[X^2] - (E[X])^2 = E[X^2]
$$

since $E[X] = 0$.

Calculating $E[X^2]$:

$$
E[X^2] = \int_{-1}^1 x^2 f_X(x) \, dx = \int_{-1}^1 x^2 \frac{2}{\pi} \sqrt{1 - x^2} \, dx
$$

Using symmetry:

$$
E[X^2] = 2 \times \frac{2}{\pi} \int_0^1 x^2 \sqrt{1 - x^2} \, dx
$$

which simplifies to:

$$
E[X^2] = \frac{4}{\pi} \int_0^1 x^2 \sqrt{1 - x^2} \, dx
$$

This integral can be evaluated using substitution:

Let $x = \sin \theta$, $dx = \cos \theta \, d\theta$, with $\theta \in [0, \pi/2]$:

$$
E[X^2] = \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \theta \times \sqrt{1 - \sin^2 \theta} \times \cos \theta \, d\theta
$$

Since $\sqrt{1 - \sin^2 \theta} = \cos \theta$:

$$
E[X^2] = \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \theta \times \cos \theta \times \cos \theta \, d\theta = \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \theta \times \cos^2 \theta \, d\theta
$$

Recall:

$$
\sin^2 \theta \times \cos^2 \theta = \frac{1}{4} \sin^2 2\theta
$$

Thus:

$$
E[X^2] = \frac{4}{\pi} \times \frac{1}{4} \int_0^{\pi/2} \sin^2 2\theta \, d\theta = \frac{1}{\pi} \int_0^{\pi/2} \sin^2 2\theta \, d\theta
$$

Using the identity:

$$
\sin^2 u = \frac{1 - \cos 2u}{2}
$$

we have:

$$
E[X^2] = \frac{1}{\pi} \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} \, d\theta = \frac{1}{2\pi} \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta
$$

Compute the integral:

$$
\int_0^{\pi/2} 1 \, d\theta = \frac{\pi}{2}
$$

$$
\int_0^{\pi/2} \cos 4\theta \, d\theta = \frac{\sin 4\theta}{4} \bigg|_0^{\pi/2} = \frac{\sin 2\pi}{4} - 0 = 0
$$

Thus:

$$
E[X^2] = \frac{1}{2\pi} \times \left(\frac{\pi}{2} - 0 \right) = \frac{1}{2\pi} \times \frac{\pi}{2} = \frac{1}{4}
$$

Therefore:

$$
Var(X) = E[X^2] = \frac{1}{4}
$$

and similarly for Y.

Summary of Properties

Property	Value
$E[X]$	0
$E[Y]$	0
$Var(X)$	$\frac{1}{4}$
$Var(Y)$	$\frac{1}{4}$
Covariance $Cov(X, Y)$	0 (due to independence and symmetry)

Applications of Uniform Point Distribution in a Circle

Monte Carlo Simulations

Randomly selecting points uniformly within a circle is fundamental in Monte Carlo methods, used in numerical integration, physics simulations, and modeling natural phenomena.

Computer Graphics and Rendering

In computer graphics, uniform sampling within a circle is essential for anti-aliasing, particle effects, and texture mapping, ensuring that visual effects are evenly distributed.

Wireless Communication and Signal Propagation

Modeling the position of devices within circular coverage areas allows for analyzing signal strength, interference, and network capacity.

Physics and Material Science

Studying particle distributions within circular regions helps in understanding phenomena such as diffusion, particle collisions, and material properties.

Extending to Other Distributions and Geometric Shapes

While this article

Frequently Asked Questions

What is the significance of choosing points uniformly at random within a circle of radius 1?

Choosing points uniformly at random within a circle of radius 1 ensures that every point inside the circle has an equal probability of being selected, which is essential for unbiased statistical analysis and probabilistic modeling.

How are the coordinates X and Y distributed when a point is uniformly chosen inside a circle of radius 1?

The coordinates X and Y are jointly distributed such that the point (X, Y) is uniformly spread over the area of the circle, with their combined distribution reflecting a uniform density over the circle's interior.

What is the probability density function (pdf) of the coordinates X and Y in this scenario?

The joint pdf of X and Y is constant within the circle, specifically $1/\pi$ for points inside the circle ($x^2 + y^2 \leq 1$), and zero outside, ensuring uniform distribution over the area.

How can one generate random coordinates (X, Y) uniformly inside the circle of radius 1?

A common method is to generate a uniform random variable for the angle θ in $[0, 2\pi)$ and a radius r with the probability density proportional to r, then set $X = r \cos \theta$ and $Y = r \sin \theta$ to ensure uniformity.

What is the expected value of the coordinates X and Y for a point uniformly chosen in the circle?

Due to symmetry, the expected values E[X] and E[Y] are both zero, as the distribution is symmetric about the center of the circle.

What is the variance of X and Y for a uniformly chosen point in the circle?

The variance of X and Y is 1/2, derived from the uniform distribution over the circle, reflecting the spread of the points around the center.

How does the joint distribution of (X, Y) change if the circle's radius is increased or decreased?

The joint distribution remains uniform, but the support changes to the new circle's radius; the density adjusts accordingly to maintain a total probability of 1 over the area.

Can the distribution of the distance from the origin be derived for the points inside the circle?

Yes, the distribution of the distance $R = \sqrt{X^2 + Y^2}$ is given by the probability density function $f_R(r) = 2r$ for r in [0, 1], indicating a higher likelihood of points being farther from the center.

Why is understanding the distribution of points inside a circle important in statistical modeling?

It provides foundational insights into spatial randomness, helps in simulation studies, and is applicable in fields like physics, engineering, and computer graphics where uniform point distribution within geometric shapes is relevant.

Additional Resources

Let X And Y Denote The Coordinates Of A Point Uniformly Chosen In The Circle Of Radius 1 Centered At The Origin — this statement encapsulates a fundamental problem in probability theory and geometric analysis. It sets the stage for exploring how points are distributed within a circle, how to compute probabilities and expectations related to such points, and how to analyze their behavior under various transformations. Understanding this problem is key to numerous applications in fields ranging from computational geometry and physics to machine learning and statistical simulation.

In this comprehensive guide, we will delve into the mathematical foundations, derivations, and implications of selecting a point uniformly at random within a unit circle. We will examine the distributions of the coordinates, the joint probability density function (pdf), the marginal distributions, and expected values. Moreover, we'll explore how to compute probabilities for specific regions, analyze the geometric intuition behind these distributions, and discuss potential extensions and applications.

The Setting: Uniform Selection in a Circle

Imagine a circle centered at the origin with radius 1, described mathematically as:

$$ \text{Circle } C = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} $$

When we say "Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin," we are defining a probability space where each point inside this circle is equally likely to be selected. This uniformity implies that the probability density is constant over the area of the circle.

Why is this problem important?

- Simulation and Monte Carlo Methods: Random sampling within geometric shapes is foundational for simulation techniques.
- Geometric Probability: It provides insights into the likelihood of a point falling within certain subregions.
- Spatial Statistics: Understanding distributions of points within regions helps in modeling phenomena in physics, ecology, and engineering.
- Machine Learning: Algorithms that rely on uniform sampling or spatial distributions often assume such models.

Mathematical Foundations

1. Uniform Distribution in a Circle

The key principle is that the probability density function (pdf) must be constant over the area of the circle and zero outside. Since the total probability over the entire circle must be 1:

$$
\iint_{x^2 + y^2 \leq 1} f_{X,Y}(x, y) \, dx\, dy = 1
$$

Given the uniformity:

$$
f_{X,Y}(x, y) = c, \quad \text{for } x^2 + y^2 \leq 1
$$

The constant c is determined by the area of the circle:

$$
c \times \text{Area of circle} = 1 \quad \Rightarrow \quad c \times \pi \times 1^2 = 1 \quad \Rightarrow \quad c = \frac{1}{\pi}
$$

\]

Thus, the joint pdf is:

\[
f_{X,Y}(x, y) = \frac{1}{\pi} \quad \text{for } x^2 + y^2 \leq 1
\]

---

2. Distribution of Coordinates

Because the joint distribution is uniform over the disk, the marginal distributions of \(X \) and \(Y \) are symmetric and identical. However, their individual distributions are not uniform over the interval \([-1, 1]\). Instead, their distributions are derived from the joint density.

Deriving Marginal Distributions

1. Distribution of \(X \)

The marginal density of \(X \), denoted \(f_X(x) \), is obtained by integrating the joint density over all \(y \) such that \((x, y) \) lies within the circle:

\[
f_X(x) = \int_{-\sqrt{1 - x^2}}^{\sqrt{1 - x^2}} f_{X,Y}(x, y) \, dy
\]

Since \(f_{X,Y}(x, y) = \frac{1}{\pi} \):

\[
f_X(x) = \frac{1}{\pi} \times 2 \sqrt{1 - x^2} \quad \text{for } -1 \leq x
\leq 1
\]

This is because for each fixed \(x \), the \(y \)-values that satisfy \(
x^2 + y^2 \leq 1 \) are:

\[
y \in [-\sqrt{1 - x^2}, \sqrt{1 - x^2}]
\]

and the length of this interval is \(2 \sqrt{1 - x^2} \).

Thus,

\[
f_X(x) = \frac{2}{\pi} \sqrt{1 - x^2}, \quad -1 \leq x \leq 1

\]

Similarly, Y has the same distribution:

$$
f_Y(y) = \frac{2}{\pi} \sqrt{1 - y^2}
$$

---

2. Distribution of R (Radius)

Often, it is more convenient to analyze the distribution of the distance from the origin, $R = \sqrt{X^2 + Y^2}$. Since the points are uniformly distributed in the circle, what is the distribution of R?

The cumulative distribution function (CDF) of R, $F_R(r)$, is:

$$
F_R(r) = P(R \leq r) = \frac{\text{Area of the disk of radius } r}{\text{Area of the circle of radius 1}} = \frac{\pi r^2}{\pi} = r^2
$$

for $0 \leq r \leq 1$.

Differentiating this gives the probability density function:

$$
f_R(r) = \frac{d}{dr} F_R(r) = 2r, \quad 0 \leq r \leq 1
$$

This indicates that the distribution of the radius R is not uniform, but instead increases linearly with r.

Geometric Intuition and Visualization

Understanding the distributions requires visualizing how points are spread within the circle.

- Coordinate Distribution: The density of X (or Y) peaks at the center ($x = 0$) and diminishes toward the edges ($x = \pm 1$). The shape $\sqrt{1 - x^2}$ is a semi-ellipse, reflecting the circular symmetry.

- Radius Distribution: The probability of a point being at a distance r from the origin is proportional to r, meaning points are more likely to be found near the boundary than near the center, despite the uniformity in the two-dimensional plane.

Computing Probabilities and Expectations

1. Probability of Falling Within a Subregion

Suppose you want to compute the probability that a randomly chosen point lies within a smaller circle of radius r_0:

$$
P(R \leq r_0) = r_0^2
$$

since from the cumulative distribution:

$$
F_R(r_0) = r_0^2
$$

2. Expected Values

Expected X and Y:

By symmetry, both have zero mean:

$$
E[X] = E[Y] = 0
$$

Variance of X:

$$
Var(X) = E[X^2] - (E[X])^2 = E[X^2]
$$

Calculating $E[X^2]$:

$$
E[X^2] = \int_{-1}^1 x^2 f_X(x) \, dx = \int_{-1}^1 x^2 \times \frac{2}{\pi} \sqrt{1 - x^2} \, dx
$$

Use symmetry:

$$
E[X^2] = 2 \times \int_{0}^1 x^2 \times \frac{2}{\pi} \sqrt{1 - x^2} \, dx
$$

which simplifies to:

\[

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E[X^2] = \frac{4}{\pi} \int_0^1 x^2 \sqrt{1 - x^2} \, dx
\]

Substitute \( x = \sin \theta \), \( dx = \cos \theta \, d\theta \):

\[
x = \sin \theta, \quad \theta \in [0, \pi/2]
\]

then:

\[
E[X^2] = \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \theta \times \sqrt{1 - \sin^2
\theta} \times \cos \theta \, d\theta
\]

Since \( \sqrt{1 - \sin^2 \theta} = \cos \theta \):

\[
E[X^2] = \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \theta \times \cos \theta \times
\cos \theta \, d\theta = \frac{4}{\pi} \int_0^{\pi/2} \
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