Let X Be An Age-at-death Random Variable With Mortality Rate (x)=2kx For X0, And Where K>0. The Mean

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Understanding mortality rates and their implications is fundamental in actuarial science, demography, and related fields. When analyzing age-at-death distributions, it's crucial to model the underlying stochastic processes accurately. In this context, consider a random variable X representing the age at death of an individual, where the mortality (hazard) rate at age x is given by $\mu(x) = 2kx$ for $x \geq X_0$, with $K > 0$. This model suggests that the risk of death increases linearly with age, scaled by a constant factor $2k$. Our goal here is to derive the mean (expected age at death) of this distribution, explore its properties, and understand its implications.

Understanding the Mortality Rate Function

Definition of the Hazard (Mortality) Rate

The hazard rate $\mu(x)$ describes the instantaneous risk of death at age x, given survival up to that age. Formally, for a non-negative random variable X:

$$
\mu(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta x \mid X \geq x)}{\Delta x}
$$

In our case:

$$
\mu(x) = 2k x, \quad x \geq X_0
$$

where X_0 is the starting age (possibly zero), and $k > 0$ controls the rate at which mortality increases with age.

Implications of the Linear Hazard Rate

The linear form $\mu(x) = 2k x$ indicates that the risk of death accelerates with age, mimicking real-world aging processes. At younger ages, the mortality is lower, but it increases as age advances. This model is analogous to a quadratic form in survival analysis, often associated with certain types of Weibull or gamma distributions.

Deriving the Survival Function

Relationship Between Hazard Rate and Survival Function

The survival function $S(x) = P(X > x)$ relates to the hazard rate via:

$$
S(x) = \exp\left(- \int_{X_0}^{x} \mu(t) dt \right)
$$

Given $\mu(t) = 2k t$, the integral becomes:

$$
\int_{X_0}^{x} 2k t dt = k (x^2 - X_0^2)
$$

assuming $X_0 \ge 0$. Therefore, the survival function is:

$$
\boxed{
S(x) = \exp\left(-k (x^2 - X_0^2) \right), \quad x \geq X_0
}
$$

For $x < X_0$, the survival function is 1 if the process starts at X_0.

Probability Density Function (pdf) of X

The pdf is derived from the survival function:

$$
f(x) = - \frac{d}{dx} S(x) = \mu(x) S(x)
$$

Thus,

$$
f(x) = 2k x \exp\left(-k (x^2 - X_0^2) \right), \quad x \geq X_0
$$

This density characterizes the distribution of age at death.

Calculating the Mean Age at Death

Expected Value Definition

The mean or expected age at death $E[X]$ is:

$$
E[X] = \int_{X_0}^{\infty} x f(x) dx
$$

Substituting $f(x)$:

$$
E[X] = \int_{X_0}^{\infty} x \cdot 2k x \exp\left(-k (x^2 - X_0^2) \right) dx = 2k \int_{X_0}^{\infty} x^2 \exp\left(-k (x^2 - X_0^2) \right) dx
$$

Simplify the exponential:

$$
E[X] = 2k \int_{X_0}^{\infty} x^2 e^{-k x^2} e^{k X_0^2} dx
$$

Note that $e^{k X_0^2}$ is a constant with respect to x, so:

$$
E[X] = 2k e^{k X_0^2} \int_{X_0}^{\infty} x^2 e^{-k x^2} dx
$$

Evaluating the Integral for the Expected Value

Changing Variables

Let:

$$
t = x \sqrt{k} \Rightarrow x = \frac{t}{\sqrt{k}}, \quad dx = \frac{dt}{\sqrt{k}}
$$

When $x = X_0$, $t = X_0 \sqrt{k}$. The integral becomes:

$$
E[X] = 2k e^{k X_0^2} \int_{t = X_0 \sqrt{k}}^{\infty} \left(\frac{t}{\sqrt{k}} \right)^2 e^{-t^2} \frac{dt}{\sqrt{k}}
$$

\]

Simplify:

\[
E[X] = 2k e^{k X_0^2} \frac{1}{k^{3/2}} \int_{X_0 \sqrt{k}}^{\infty} t^2 e^{-t^2} dt
\]

\[
E[X] = 2 e^{k X_0^2} \frac{1}{\sqrt{k}} \int_{X_0 \sqrt{k}}^{\infty} t^2 e^{-t^2} dt
\]

Evaluating the Integral \(\int_{a}^{\infty} t^2 e^{-t^2} dt \)

This integral is a standard form. Recall:

\[
\int_{a}^{\infty} t^2 e^{-t^2} dt = \frac{\sqrt{\pi}}{4} \operatorname{erfc}(a) - \frac{a}{2} e^{-a^2}
\]

where \(\operatorname{erfc}(a) \) is the complementary error function.

Applying this:

\[
E[X] = 2 e^{k X_0^2} \frac{1}{\sqrt{k}} \left[\frac{\sqrt{\pi}}{4} \operatorname{erfc}(X_0 \sqrt{k}) - \frac{X_0 \sqrt{k}}{2} e^{-(X_0 \sqrt{k})^2} \right]
\]

Note that:

\[
e^{-(X_0 \sqrt{k})^2} = e^{-k X_0^2}
\]

and \(e^{k X_0^2} \times e^{-k X_0^2} = 1 \). Therefore, the expected age simplifies to:

\[
\boxed{
E[X] = \frac{\sqrt{\pi}}{2 \sqrt{k}} \operatorname{erfc}(X_0 \sqrt{k}) - X_0
}
\]

Final Expression for the Mean Age at Death

Putting all parts together, the mean age at death for the distribution with hazard rate $\mu(x) = 2k x$ starting from age X_0 is:

$$
\boxed{
\textbf{Mean}:\quad E[X] = \frac{\sqrt{\pi}}{2 \sqrt{k}} \operatorname{erfc}(X_0 \sqrt{k}) - X_0
}
$$

where:

- $\operatorname{erfc}(\cdot)$ is the complementary error function,
- $X_0 \geq 0$,
- $K > 0$ controls the rate at which mortality increases.

Insights and Implications of the Result

Effect of Parameter K

- As K increases, $\sqrt{k}$ increases, reducing $E[X]$. This indicates higher mortality rates lead to a lower expected age at death.
- Conversely, smaller K values extend the expected lifespan.

Impact of Starting Age X_0

- For $X_0 = 0$, the expression simplifies to:

$$
E[X] = \frac{\sqrt{\pi}}{2 \sqrt{k}} \operatorname{erfc}(0) = \frac{\sqrt{\pi}}{2 \sqrt{k}} \times 1 = \frac{\sqrt{\pi}}{2 \sqrt{k}}
$$

- As X_0 increases, \(\operatorname{

Frequently Asked Questions

What is the probability density function (PDF) of the age-at-

death random variable X given the mortality rate μ(x) = 2kx?

The PDF of X is f(x) = 2k e^{-k x^2} for x ≥ 0, derived from the hazard function μ(x) and the survival function S(x) = e^{-k x^2}.

How do we compute the expected age at death E[X] for the given mortality rate μ(x) = 2kx?

E[X] is calculated as E[X] = $\int_0$^∞ x f(x) dx, which evaluates to (1/√k) × (√π/2), based on the properties of the Rayleigh distribution.

What distribution does the age-at-death variable X follow with the given mortality rate μ(x) = 2kx?

X follows a Rayleigh distribution with scale parameter σ = 1/√(2k).

How does changing the parameter k affect the mean age at death in this model?

The mean age at death is inversely proportional to √k; increasing k decreases the mean, indicating higher mortality rates lead to shorter lifespans.

What is the significance of the parameter k in modeling mortality rates for X?

Parameter k determines the rate at which mortality increases with age; a higher k signifies a faster increase in mortality, affecting the shape of the distribution.

Can the mean age at death be expressed in terms of k explicitly, and if so, what is the formula?

Yes, the mean age at death is E[X] = (√π)/(2√k), explicitly showing its dependence on k.

Additional Resources

Let X Be An Age-at-death Random Variable With Mortality Rate (x) = 2kx for X_0, And Where K > 0. The Mean

Understanding the nuances of mortality modeling has long been a critical pursuit in actuarial science, demography, and public health. When we describe the age at which individuals die using a stochastic variable—say, X—it's essential to delve into the mathematical structure underpinning such models. One particularly intriguing case arises when the mortality rate at age x, denoted as μ(x), follows a linear form: μ(x) = 2kx, where k is a positive constant. This model not only provides insights into the nature of aging and mortality but also allows for the explicit calculation of crucial statistical measures, such as the mean age at death.

In this article, we explore the properties of this specific mortality model, focusing on deriving the expected age at death, or the mean of the distribution of X. We'll examine the assumptions underlying the model, the mathematical derivations involved, and the implications these have for understanding real-world mortality patterns.

Understanding the Mortality Rate Model: The Foundation

Defining the Mortality Rate Function

The core of any mortality model lies in the hazard or mortality rate function, denoted as $\mu(x)$. This function represents the instantaneous risk of death at age x, given survival up to that age. For our case, the model assumes:

$\mu(x) = 2kx$, where $k > 0$

This is a linear function of age x, implying that the risk of death increases proportionally with age. Such a model captures a fundamental biological insight: as individuals age, their risk of mortality tends to rise, often in a manner that can be approximated as linear over certain age ranges.

Implications of the Model Assumption

This assumption simplifies the complex reality of mortality, which often involves multiple phases (e.g., infant mortality, middle-aged stability, old-age decline). Yet, the linear form $\mu(x) = 2kx$ provides a tractable framework to derive analytical expressions for the distribution of age at death. It aligns with the Gompertz law in demography, which states that mortality rates increase exponentially with age, especially in adult populations. Here, the linear form is a simplified approximation that captures the early to middle age increase in mortality.

Mathematical Framework: From Mortality Rate to Survival Function

The Relationship Between Mortality Rate and Survival Function

The survival function, S(x), denotes the probability that an individual survives beyond age x. It is intimately related to the mortality rate $\mu(x)$ via the hazard function:

$S(x) = \exp(-\int_0^x \mu(t)\, dt)$

This integral accumulates the hazard over age, translating it into the probability of survival.

Calculating the Survival Function for Our Model

Given $\mu(x) = 2kx$, the integral becomes:

$\int_0^x 2kt\, dt = 2k \int_0^x t\, dt = 2k (x^2 / 2) = k x^2$

Thus, the survival function simplifies to:

$S(x) = \exp(-k x^2)$

This is a key result: it indicates that the probability of surviving beyond age x decreases exponentially with the square of age, governed by the parameter k.

Probability Density Function of Age at Death

The density function f(x), describing the likelihood that death occurs precisely at age x, is derived from the survival function:

$f(x) = \mu(x) S(x) = 2kx \exp(-k x^2)$

This function, known as the age-at-death probability density, characterizes the distribution of the random variable X.

Deriving the Mean Age at Death

Formulating the Expectation

The mean or expected age at death, denoted E[X], is given by:

$E[X] = \int_0^\infty x f(x)\, dx = \int_0^\infty x\, 2kx \exp(-k x^2)\, dx$

Simplifying the integrand:

$E[X] = 2k \int_0^\infty x^2 \exp(-k x^2)\, dx$

Calculating this integral involves recognizing a standard form related to the Gamma function or the Gaussian integral.

Evaluating the Integral

The integral:

$I = \int_0^\infty x^2 \exp(-a x^2) dx$

is a well-known integral, with the solution:

$I = (1/2) \sqrt{\pi} / a^{3/2}$

where $a > 0$.

In our case, $a = k$, so:

$\int_0^\infty x^2 \exp(-k x^2) dx = (1/2) \sqrt{\pi} / k^{3/2}$

Substituting back into the expression for E[X]:

$E[X] = 2k (1/2) \sqrt{\pi} / k^{3/2} = \sqrt{\pi} k / k^{3/2} = \sqrt{\pi} / \sqrt{k}$

Thus, the expected age at death simplifies to:

$E[X] = \sqrt{\pi} / \sqrt{k}$

Interpretation of the Result

The mean age at death is inversely proportional to the square root of the parameter k. If k increases (implying a steeper rise in mortality with age), the expected age at death decreases. Conversely, smaller k values suggest longer average lifespans, aligning with the intuition that higher mortality rates reduce average life expectancy.

Broader Implications and Applications

Modeling Mortality in Populations

While the model $\mu(x) = 2kx$ is simplified, it offers a foundation for understanding how increasing age impacts mortality risk. It can be adapted or extended to fit real-world data, assisting demographers and public health officials in projecting life expectancy, planning healthcare resources, or analyzing the effects of interventions.

Limitations and Extensions

- Simplification of Mortality Patterns: Real mortality involves multiple phases; thus, more complex models like the Gompertz or Makeham laws incorporate additional parameters.
- Age Range Validity: The linear model may only be appropriate over certain age ranges; biological realities often demand more nuanced functions.
- Parameter Estimation: Estimating k from empirical data involves statistical techniques like maximum likelihood estimation, which can refine the model's accuracy.

Potential for Further Research

Researchers might explore:

- How modifications to $\mu(x)$ influence the distribution of X.
- Extensions to include stochastic effects or environmental factors.
- Comparing this model's predictions with demographic data across different populations.

Conclusion

The exploration of an age-at-death random variable with a mortality rate $\mu(x) = 2kx$ reveals elegant mathematical properties and offers valuable insights into mortality dynamics. The derivation of the mean age at death, $E[X] = \sqrt{\pi} / \sqrt{k}$, underscores the direct relationship between the risk of mortality and expected lifespan. While simplified, this model serves as a stepping stone toward more sophisticated mortality analyses, enhancing our understanding of aging and longevity. As demographic data continues to grow richer and more complex, such foundational models will remain vital tools for scientists, policymakers, and actuaries working to comprehend and influence human life expectancy.

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