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Introduction

In the realm of calculus, differential equations serve as powerful tools for modeling a wide array of natural and engineered phenomena. Among these, first-order differential equations—particularly those involving functions and their derivatives—are fundamental in understanding how quantities change relative to each other.

The given problem involves a function $Y = F(x)$, which is twice-differentiable, meaning it has continuous derivatives up to the second order. The initial condition $F(1) = 2$ provides a specific point on the function, and the differential equation $\frac{dy}{dx} = y^3 + 3$ governs the relationship between the function Y and the independent variable x.

This scenario prompts a classic differential equation problem: how to find the explicit form of $F(x)$ given the differential equation and initial condition? Furthermore, the question "What is the value?" hints at determining the specific value of $F(x)$ at a particular point, or perhaps solving for a particular constant or parameter within the solution.

In this comprehensive article, we will explore the methods for solving such differential equations, analyze the given initial conditions, and illustrate step-by-step procedures to find the explicit form of $F(x)$. We will also discuss the significance of the solution in practical contexts and how to interpret the results.

Understanding the Differential Equation $\frac{dy}{dx} = y^3 + 3$

Nature of the Differential Equation

The differential equation:

$$
\frac{dy}{dx} = y^3 + 3
$$

is a first-order ordinary differential equation that is nonlinear due to the y^3 term. Its structure suggests that it might be solvable through separation of variables, substitution, or an integrating factor, depending on the approach.

Characteristics of the Equation

- Type: Nonlinear first-order ODE
- Order: First order
- Variables: x (independent variable), y (dependent variable)
- Initial condition: $F(1) = 2$

Solving the Differential Equation

Step 1: Recognize the Form

The differential equation can be written as:

$$
\frac{dy}{dx} = y^3 + 3
$$

which suggests that it is separable because the right side is a function of y only, and the derivative is with respect to x. So, we can write:

$$
\frac{dy}{y^3 + 3} = dx
$$

Step 2: Integrate Both Sides

The goal is to integrate both sides:

$$
\int \frac{dy}{y^3 + 3} = \int dx
$$

The integral on the left involves rational functions with a cubic polynomial in the denominator, which can be tackled using substitution methods.

Partial Fraction Decomposition

To integrate:

$$
\int \frac{dy}{y^3 + 3}
$$

we first factor or decompose the denominator if possible.

Factorizing $y^3 + 3$

Note that:

$$
y^3 + 3 = y^3 + 1^3 + 2
$$

But this is not a sum of cubes directly. Alternatively, observe that:

$$
y^3 + 3 = y^3 + (\sqrt[3]{3})^3
$$

which is a sum of cubes. The sum of cubes formula:

$$
a^3 + b^3 = (a + b)(a^2 - ab + b^2)
$$

Applying this:

$$
y^3 + (\sqrt[3]{3})^3 = (y + \sqrt[3]{3})(y^2 - y \sqrt[3]{3} + (\sqrt[3]{3})^2)
$$

Hence,

$$
y^3 + 3 = (y + \sqrt[3]{3})(y^2 - y \sqrt[3]{3} + (\sqrt[3]{3})^2)
$$

where:

$$
(\sqrt[3]{3})^2 = \sqrt[3]{3^2} = \sqrt[3]{9}
$$

Thus,

$$
y^3 + 3 = (y + \sqrt[3]{3})(y^2 - y \sqrt[3]{3} + \sqrt[3]{9})
$$

Now, the integral becomes:

$$
\int \frac{dy}{(y + \sqrt[3]{3})(y^2 - y \sqrt[3]{3} + \sqrt[3]{9})}
$$

Step 3: Partial Fraction Decomposition

Express the integrand as:

\[
\frac{1}{(y + a)(y^2 + p y + q)} = \frac{A}{ y + a } + \frac{B y + C}{ y^2 + p y + q }
\]

where:

- \(a = \sqrt[3]{3}\)
- \(p = - \sqrt[3]{3}\)
- \(q = \sqrt[3]{9}\)

Set up:

\[
\frac{1}{(y + a)(y^2 + p y + q)} = \frac{A}{ y + a } + \frac{B y + C}{ y^2 + p y + q }
\]

Multiply both sides by the denominator:

\[
1 = A (y^2 + p y + q) + (B y + C)(y + a)
\]

Expanding:

\[
1 = A y^2 + A p y + A q + B y (y + a) + C (y + a)
\]

which simplifies to:

\[
1 = A y^2 + A p y + A q + B y^2 + a B y + C y + a C
\]

Group like terms:

\[
1 = (A + B) y^2 + (A p + a B + C) y + (A q + a C)
\]

Matching coefficients:

- For \(y^2 \):

\[
A + B = 0 \Rightarrow B = -A
\]

- For \(y \):

\[

$$
A p + a B + C = 0
$$

- For the constant term:

$$
A q + a C = 1
$$

Substitute $B = -A$:

$$
A p + a (-A) + C = 0 \Rightarrow A p - a A + C = 0 \Rightarrow C = A (a - p)
$$

Similarly, from the constant term:

$$
A q + a C = 1
$$

Substitute $C = A (a - p)$:

$$
A q + a \times A (a - p) = 1
$$

Factor out A:

$$
A (q + a (a - p)) = 1
$$

Now, plug in the known values:

- $a = \sqrt[3]{3}$
- $p = - \sqrt[3]{3}$
- $q = \sqrt[3]{9}$

Calculate $a - p$:

$$
a - p = \sqrt[3]{3} - (- \sqrt[3]{3}) = 2 \sqrt[3]{3}
$$

Calculate $q + a (a - p)$:

$$
\sqrt[3]{9} + \sqrt[3]{3} \times 2 \sqrt[3]{3}
$$

Note that:

$$
\sqrt[3]{3} \times \sqrt[3]{3} = \sqrt[3]{3^2} = \sqrt[3]{9}
$$

So,

$$
a (a - p) = \sqrt[3]{3} \times 2 \sqrt[3]{3} = 2 \times \sqrt[3]{3} \times \sqrt[3]{3} = 2 \times \sqrt[3]{9}
$$

Therefore,

$$
q + a (a - p) = \sqrt[3]{9} + 2 \sqrt[3]{9} = 3 \sqrt[3]{9}
$$

Putting it all together:

$$
A \times 3 \sqrt[3]{9} = 1 \Rightarrow A = \frac{1}{3 \sqrt[3]{9}}
$$

Similarly,

$$
C = A (a - p) = \frac{1}{3 \sqrt[3]{9}} \times 2 \sqrt[3]{3} = \frac{2 \sqrt[3]{3} }{3 \sqrt[3]{9}}
$$

Note that:

$$
\sqrt[3]{9} = \sqrt[3]{3^2} = (\sqrt[3]{3})^2
$$

So,

$$
C = \frac{2 \sqrt[3]{3} }{3 (\sqrt[3]{3})^2 } = \frac{2}{3 \sqrt[3]{3} }
$$

Frequently Asked Questions

Given the function $Y = F(x)$ with $F(1) = 2$ and the differential equation $dy/dx = y^3 + 3$, what is the value of $F(1)$?

The value of $F(1)$ is given as 2, so $F(1) = 2$.

How do you solve the differential equation $dy/dx = y^3 + 3$ with initial condition $F(1) = 2$?

You can separate variables: $dy/(y^3 + 3) = dx$, then integrate both sides and apply the initial condition to find $F(x)$.

What is the significance of the initial condition $F(1) = 2$ in solving the differential equation?

It helps determine the constant of integration after integrating the differential equation, ensuring a unique solution passing through the point (1, 2).

Is the function $F(x)$ increasing or decreasing near $x=1$ based on $dy/dx = y^3 + 3$?

Since $y^3 + 3 > 0$ for all real y, the derivative dy/dx is always positive, indicating that $F(x)$ is increasing near $x=1$.

What are possible methods to find an explicit expression for $F(x)$ given $dy/dx = y^3 + 3$ and initial condition $F(1) = 2$?

Methods include separation of variables and integration, possibly involving substitution or partial fractions, to solve the differential equation explicitly.

Additional Resources

Let $Y = F(x)$ Be A Twice-differentiable Function Such That $F(1) = 2$ And Dydx = $y^3 + 3$. What Is The Value?

This problem explores a fundamental aspect of differential equations, involving the determination of a particular function $F(x)$ given its derivative in terms of itself. The challenge lies in solving a nonlinear first-order differential equation where the derivative $\frac{dy}{dx}$ depends explicitly on the function y, which itself is a function of x. Such problems are core to understanding dynamic systems, modeling physical phenomena, and mathematical analysis, making this a valuable exercise for students and practitioners alike.

Understanding the Problem Statement

The problem states: Let $Y = F(x)$ be a twice-differentiable function such that $F(1) = 2$ and $\frac{dy}{dx} = y^3 + 3$. The question asks: what is the value?

At first glance, the phrase "What is the value?" appears ambiguous. It could refer to several possibilities: the value of $F(x)$ at a certain point, the value of the derivative, or perhaps a specific constant in the solution. Given the context, the most logical interpretation is that it seeks the particular solution $F(x)$, or perhaps the value of $F(x)$ at a specific point, or the general solution to the differential equation satisfying the initial condition $F(1)=2$.

Since the initial condition $F(1) = 2$ is provided, the primary goal becomes solving the differential equation:
$$
\frac{dy}{dx} = y^3 + 3
$$
with the initial condition $y(1) = 2$.

Breaking Down the Differential Equation

Type of Equation

The differential equation
$$
\frac{dy}{dx} = y^3 + 3
$$
is a first-order nonlinear ordinary differential equation. Its nonlinearity stems from the y^3 term, which prevents straightforward linear solution methods.

Method of Solution

Given its form, the most suitable approach is separation of variables, which involves rewriting the equation so that all y terms are on one side and all x terms on the other.

Solving the Differential Equation

Separation of Variables

Rewrite the equation as:
$$
\frac{dy}{y^3 + 3} = dx
$$
This form enables integration on both sides. The challenge is to integrate $\frac{1}{y^3 + 3}$ with respect to y.

Integral of $\frac{1}{y^3 + 3}$

The integral
$$
\int \frac{dy}{y^3 + 3}
$$
requires partial fraction decomposition or substitution. First, factor the denominator if possible.

Note that:
$$
y^3 + 3 = y^3 + 3
$$
has no real linear factors, but we can factor it over complex numbers or use substitution techniques. Alternatively, we can attempt to factor it over real numbers using substitution:

Let’s observe that
$$
y^3 + 3 = y^3 + 3
$$
which can be viewed as a sum of cubes:
$$
a^3 + b^3 = (a + b)(a^2 - ab + b^2)
$$
with $a=y$ and $b= \sqrt[3]{3}$.

Since 3 is a constant, and $3 = (\sqrt[3]{3})^3$,
we have:
$$
y^3 + (\sqrt[3]{3})^3 = (y + \sqrt[3]{3}) \left(y^2 - y \sqrt[3]{3} + (\sqrt[3]{3})^2 \right)
$$

Thus,
$$
\int \frac{dy}{(y + \sqrt[3]{3})(y^2 - y \sqrt[3]{3} + (\sqrt[3]{3})^2)}
$$

This partial fraction decomposition allows splitting the integral into manageable parts.

Partial Fraction Decomposition

Express:
$$
\frac{1}{(y + a)(y^2 - a y + a^2)} = \frac{A}{y + a} + \frac{By + C}{ y^2 - a y + a^2}
$$
where $a = \sqrt[3]{3}$.

Multiplying both sides by the denominator gives:
$$
1 = A (y^2 - a y + a^2) + (By + C)(y + a)
$$
Expanding:
$$
1 = A y^2 - A a y + A a^2 + By^2 + B a y + C y + C a
$$
Group similar terms:
$$
1 = (A + B) y^2 + (-A a + B a + C) y + (A a^2 + C a)
$$

Matching coefficients with the constant 1:
$$
\begin{cases}
A + B = 0 \\
-A a + B a + C = 0 \\
A a^2 + C a = 1
\end{cases}
$$

Solve for A, B, C:
From the first: $B = -A$
Substitute into the second:
$$
-A a - A a + C = 0 \Rightarrow -2 A a + C = 0 \Rightarrow C = 2 A a
$$
From the third:
$$
A a^2 + C a = 1
$$
Substitute C:
$$
A a^2 + (2 A a) a = 1 \Rightarrow A a^2 + 2 A a^2 = 1 \Rightarrow 3 A a^2 = 1
$$
Solve for A:
$$
A = \frac{1}{3 a^2}
$$

Now, find C:

\[
C = 2 A a = 2 \times \frac{1}{3 a^2} \times a = \frac{2}{3 a}
\]

And \(B = -A = - \frac{1}{3 a^2}\).

Integrating Term-by-Term

Now, rewrite the integral:
\[
\int \frac{dy}{(y + a)(y^2 - a y + a^2)} = \int \left(\frac{A}{y + a} + \frac{By + C}{ y^2 - a y + a^2} \right) dy
\]
with known constants:

\[
A = \frac{1}{3 a^2}, \quad B = - \frac{1}{3 a^2}, \quad C = \frac{2}{3 a}
\]

Thus:
\[
\int \left(\frac{1}{3 a^2 (y + a)} + \frac{ - \frac{1}{3 a^2} y + \frac{2}{3 a} }{ y^2 - a y + a^2 } \right) dy
\]

Split into two integrals:

\[
I_1 = \frac{1}{3 a^2} \int \frac{dy}{y + a}
\]
and

\[
I_2 = \int \frac{ - \frac{1}{3 a^2} y + \frac{2}{3 a} }{ y^2 - a y + a^2 } dy
\]

Integral \(I_1\)

\[
I_1 = \frac{1}{3 a^2} \ln | y + a | + C
\]

Integral I_2

Note that the quadratic in the denominator can be completed to a square:

$$
y^2 - a y + a^2 = \left(y - \frac{a}{2} \right)^2 + \frac{3 a^2}{4}
$$

Let $u = y - \frac{a}{2}$, then $dy = du$. The numerator becomes:

$$
- \frac{1}{3 a^2} y + \frac{2}{3 a}
$$

Express y in terms of u:

$$
y = u + \frac{a}{2}
$$

Substitute into numerator:

$$
- \frac{1}{3 a^2} \left(u + \frac{a}{2} \right) + \frac{2}{3 a}
$$

Simplify:

\[
- \frac{u}{3 a^2} - \frac{a/2}{3 a^2} + \frac{2}{3 a} = - \frac{u}{3 a

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