LetX And Y Be Independent Poisson Random Variables With Respectivemeans Lambda1 And Lambda 2 .Calculate

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Understanding the behavior of Poisson random variables is fundamental in probability theory and statistics, especially in modeling count data such as the number of events occurring within a fixed interval or space. When dealing with two independent Poisson random variables, X and Y, each characterized by their respective means, Lambda1 and Lambda2, it becomes essential to understand how their joint behavior manifests and how to perform calculations involving their distributions.

This article aims to provide a comprehensive explanation of the properties, calculations, and applications related to independent Poisson random variables, specifically focusing on the computation of probabilities, expectations, variances, and their joint distributions. We will explore the foundational concepts, formulas, and practical examples to equip you with a thorough understanding of the topic.

Introduction to Poisson Distributions

What Is a Poisson Distribution?

The Poisson distribution is a discrete probability distribution that models the number of times an event occurs within a fixed interval or space, assuming these events happen independently and at a constant average rate. It is widely used in fields such as telecommunications, biology, insurance, and physics.

The probability mass function (PMF) of a Poisson random variable X, with mean (or rate) Lambda, is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

for $k = 0, 1, 2, \ldots$, where:
- $\lambda > 0$ is the average number of events in the interval.
- $k!$ is the factorial of k.

Properties of Poisson Random Variables

Some key properties include:
- The expected value $E[X] = \lambda$.
- The variance $Var[X] = \lambda$.
- The sum of independent Poisson variables is also Poisson-distributed, with the mean equal to the sum of the individual means.

Independent Poisson Random Variables X and Y

Definition and Assumptions

Suppose X and Y are two independent Poisson random variables with means λ_1 and λ_2, respectively. Independence implies that:

$$P(X = x, Y = y) = P(X = x) \times P(Y = y)$$

for all non-negative integers x, y.

Joint Distribution

Because X and Y are independent, their joint PMF is simply the product of their individual PMFs:

$$P(X = x, Y = y) = \frac{\lambda_1^x e^{-\lambda_1}}{x!} \times \frac{\lambda_2^y e^{-\lambda_2}}{y!}$$

This joint distribution is useful when calculating probabilities involving both variables simultaneously.

Calculating Probabilities and Expectations

Joint Probability of $X = x$ and $Y = y$

Given the independence, the joint probability is straightforward:

$$P(X = x, Y = y) = \frac{\lambda_1^x e^{-\lambda_1}}{x!} \times \frac{\lambda_2^y e^{-\lambda_2}}{y!}$$

for $x, y \geq 0$.

Marginal Distributions

The marginal distributions of X and Y are their original Poisson distributions:

$$P(X = x) = \frac{\lambda_1^x e^{-\lambda_1}}{x!}$$

$$P(Y = y) = \frac{\lambda_2^y e^{-\lambda_2}}{y!}$$

Expected Values and Variances

The expected value and variance of the sum $X + Y$ are key calculations:

- Since X and Y are independent:

$$E[X + Y] = E[X] + E[Y] = \lambda_1 + \lambda_2$$

$$Var[X + Y] = Var[X] + Var[Y] = \lambda_1 + \lambda_2$$

- For individual variables:

$$E[X] = \lambda_1, \quad Var[X] = \lambda_1$$

$$E[Y] = \lambda_2, \quad Var[Y] = \lambda_2$$

Distribution of the Sum $X + Y$

Sum of Independent Poisson Variables

One of the remarkable properties of Poisson distributions is that the sum of independent Poisson random variables is also Poisson-distributed, with the mean being the sum of the individual means:

$$X + Y \sim Poisson(\lambda_1 + \lambda_2)$$

This property simplifies many calculations, especially in scenarios where total counts are of interest.

Proof Outline

The proof leverages the properties of the Poisson PMF, convolution, and the independence assumption:

\[

$$
\begin{aligned}
P(X + Y = n) &= \sum_{k=0}^n P(X = k, Y = n - k) \\
&= \sum_{k=0}^n \frac{\lambda_1^k e^{-\lambda_1}}{k!} \times \frac{\lambda_2^{n - k} e^{-\lambda_2}}{(n - k)!} \\
&= e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^n \frac{\lambda_1^k}{k!} \times \frac{\lambda_2^{n - k}}{(n - k)!} \\
&= e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}
\end{aligned}
$$

which is the PMF of a Poisson distribution with mean $\lambda_1 + \lambda_2$.

Applications of Independent Poisson Variables

Modeling and Simulation

In real-world applications, modeling events as independent Poisson variables is common in:
- Network traffic analysis (e.g., number of packets received from different sources).
- Biological studies (e.g., counts of different cell types).
- Insurance claims (e.g., claims from different policyholders).

Simulating such variables involves generating random samples from Poisson distributions with specified means, which can be achieved using various statistical software and programming languages.

Hypothesis Testing and Confidence Intervals

When analyzing data modeled by Poisson distributions, statisticians often perform hypothesis tests to determine whether observed counts deviate significantly from expected values. Confidence intervals for λ can be constructed based on the properties of the Poisson distribution.

Estimating Parameters

Given observed data, maximum likelihood estimation (MLE) can be used to estimate λ_1 and λ_2. The MLE for a Poisson mean is simply the sample mean:

$$
\hat{\lambda}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}
$$

where x_{ij} are observed counts for variable i.

Calculations: Examples and Practice

Example 1: Probability of Specific Counts

Suppose $X \sim Poisson(3)$ and $Y \sim Poisson(5)$, with independence. What is the probability that $X = 2$ and $Y = 4$?

Solution:

$$
P(X=2, Y=4) = \frac{3^2 e^{-3}}{2!} \times \frac{5^4 e^{-5}}{4!} = \left(\frac{9 e^{-3}}{2}\right) \times \left(\frac{625 e^{-5}}{24}\right)
$$

$$
= \frac{9 \times 625 \times e^{-8}}{2 \times 24} = \frac{5625 e^{-8}}{48}
$$

Final answer:

$$
P(X=2, Y=4) = \frac{5625}{48} e^{-8}
$$

Example 2: Probability that Sum Is a Certain Value

What is $P(X + Y = 7)$?

Solution:

Since $X + Y \sim Poisson(\lambda_1 + \lambda_2)$, here $\lambda_1=3$, $\lambda_2=5$, so:

$$
P(X + Y=7) = \frac{(3+5)^7 e^{-(3+5)}}{7!} = \frac{8^7 e^{-8}}{5040}
$$

Calculating:

$$
8^7 = 8^7 = 2097152
$$

Therefore:

$$
P(X + Y=7) = \frac{2097152 e^{-8}}{5040}
$$

Advanced

Frequently Asked Questions

What is the distribution of the sum of two independent Poisson random variables X and Y with means λ_1 and λ_2?

The sum $Z = X + Y$ is also a Poisson random variable with mean $\lambda_1 + \lambda_2$.

How do you compute the probability $P(X = k)$ for a Poisson random variable X with mean λ_1?

$P(X = k) = (e^{-\lambda_1} \lambda_1^k) / k!$, for $k = 0, 1, 2, \ldots$

Given two independent Poisson variables X and Y, how do you find the joint probability $P(X = x, Y = y)$?

Since X and Y are independent, $P(X = x, Y = y) = P(X = x)$ P(Y

$= y) = [(e^{-\lambda 1} \lambda 1^x) / x!] [(e^{-\lambda 2} \lambda 2^y) / y!]$.

What is the expected value of the product E[XY] for independent Poisson variables X and Y?

Since X and Y are independent, $E[XY] = E[X] E[Y] = \lambda 1 \lambda 2$.

How would you compute the covariance between X and Y when they are independent Poisson variables?

The covariance $Cov(X, Y) = E[XY] - E[X]E[Y] = 0$, since independent variables have zero covariance.

Additional Resources

Let X and Y be independent Poisson random variables with respective means λ_1 and λ_2. Calculate

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Introduction

In the realm of probability theory and statistical modeling, the Poisson distribution holds a prominent position due to its versatility in modeling count data and rare events. When analyzing two such variables, understanding their joint behavior, especially under conditions of independence, becomes essential for both theoretical insights and practical applications. In particular, the task of calculating various probabilistic properties, moments, and joint functions of independent Poisson variables provides foundational tools for statisticians and data scientists alike.

This article aims to explore the properties of independent Poisson random variables X and Y, with means λ_1 and λ_2, respectively. We will dissect the process of calculating probabilities, moments, joint distributions, and functions involving these variables. Our approach emphasizes clarity, thoroughness, and analytical rigor, ensuring that readers gain a comprehensive understanding of the topic.

Background on Poisson Random Variables

Definition and Properties

A Poisson random variable X is characterized by its probability mass function (pmf):

$$
P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots
$$

where $\lambda > 0$ is the mean number of events in a fixed interval or space.

Key Properties:
- Mean: $\mathbb{E}[X] = \lambda$
- Variance: $\text{Var}(X) = \lambda$
- Memoryless-like property: The Poisson process models the number of events occurring independently over disjoint intervals.

Independence of Poisson Variables

Two Poisson variables X and Y are said to be independent if the joint probability factors as:

$$
P(X = x, Y = y) = P(X = x) P(Y = y)
$$

which simplifies many calculations, especially for expectations and variances of functions involving both variables.

Joint Distribution of Independent Poisson Variables

Derivation of the Joint pmf

Given the independence assumption:

$$
P(X = x, Y = y) = P(X = x) P(Y = y) = \frac{e^{-\lambda_1} \lambda_1^x}{x!} \times \frac{e^{-\lambda_2} \lambda_2^y}{y!}
$$

Thus, the joint pmf is:

$$
\boxed{
P(X = x, Y = y) = e^{-(\lambda_1 + \lambda_2)} \frac{\lambda_1^x}{x!} \frac{\lambda_2^y}{y!}
}
$$

for non-negative integers $x, y \geq 0$.

Implications of the Joint Distribution

The joint distribution being a product of individual pmfs signifies that the counts of the two processes are statistically independent. This property simplifies many subsequent calculations involving sums, conditional probabilities, and joint moments.

Calculations Involving X and Y

The core of the analysis revolves around calculating various probabilistic quantities of interest, including joint probabilities, marginal and conditional distributions, moments, and functions like sums and products of the variables.

1. Marginal Distributions

Since X and Y are independent, their marginal pmfs are straightforward:

$$
P(X = x) = \frac{e^{-\lambda_1} \lambda_1^x}{x!}
$$

\]
\[
P(Y = y) = \frac{e^{-\lambda_2} \lambda_2^y}{y!}
\]

which are simply the Poisson distributions with parameters \(\lambda_1\) and \(\lambda_2\), respectively.

2. Joint and Conditional Distributions

- Joint pmf:

\[
P(X = x, Y = y) = e^{-(\lambda_1 + \lambda_2)} \frac{\lambda_1^x}{x!} \frac{\lambda_2^y}{y!}
\]

- Conditional pmf of \(X\) given \(Y = y\):

\[
P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{\frac{e^{-\lambda_1} \lambda_1^x}{x!} \times \frac{e^{-\lambda_2} \lambda_2^y}{y!}}{\frac{e^{-\lambda_2} \lambda_2^y}{y!}} = \frac{e^{-\lambda_1} \lambda_1^x / x!}{e^{-\lambda_1}} = \frac{\lambda_1^x}{x!}

\]

which is Poisson with parameter λ_1. Similarly:

$$
P(Y = y \mid X = x) = \frac{\lambda_2^y}{y!}
$$

This confirms that the variables are independent; the conditioning does not alter the distribution.

Expectation and Moments

Understanding the moments of the variables and their functions provides insight into their behavior and variability.

1. Expectations and Variances

Since X and Y are Poisson:

$$
\mathbb{E}[X] = \lambda_1, \quad \text{Var}(X) = \lambda_1
$$

$$
\mathbb{E}[Y] = \lambda_2, \quad \text{Var}(Y) = \lambda_2
$$

2. Expectations of Sums and Products

- Sum of the variables:

$$
S = X + Y
$$

Since the sum of independent Poisson variables is also Poisson:

$$
S \sim \text{Poisson}(\lambda_1 + \lambda_2)
$$

and

$$
\mathbb{E}[S] = \lambda_1 + \lambda_2, \quad \text{Var}(S) = \lambda_1 + \lambda_2
$$

- Product XY:

$$
\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \lambda_1 \lambda_2
$$

since X and Y are independent.

- Covariance:

$$
\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0
$$

which confirms independence.

Generating Functions and Their Applications

Generating functions serve as powerful tools to analyze sums, moments, and distributions.

1. Probability Generating Function (PGF)

For a Poisson random variable X:

$$
G_X(t) = \mathbb{E}[t^X] = \exp(\lambda_1 (t - 1))
$$

Similarly, for Y:

$$
G_Y(t) = \exp(\lambda_2 (t - 1))
$$

2. Joint Generating Function

Given independence:

$$
G_{X,Y}(s, t) = \mathbb{E}[s^X t^Y] = G_X(s) G_Y(t) = \exp(\lambda_1 (s - 1)) \times \exp(\lambda_2 (t - 1))
$$

This joint PGF facilitates derivation of moments and joint probabilities.

Calculations of Specific Probabilistic Quantities

A. Probability that \(X\) and \(Y\) are both zero

\[
P(X=0, Y=0) = e^{-(\lambda_1 + \lambda_2)}
\]

B. Probability that the sum equals a specific value

\[
P(S = n) = \frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^n}{n!}
\]

since the sum is Poisson distributed with mean \(\lambda_1 + \lambda_2\).

C. Expected value of the product \(XY\)

Since the variables are independent:

\[

$$
\mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y] = \lambda_1 \lambda_2
$$

D. Covariance and correlation

- Covariance:

$$
\text{Cov}(X, Y) = 0
$$

- Correlation coefficient:

$$
\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = 0
$$

which emphasizes their independence.

Practical Applications of These Calculations

Understanding these calculations is critical in a variety of fields such as:

- Epidemiology: Modeling the number of disease cases across different regions.
- Network Traffic: Counting packet arrivals over different network segments.
- Manufacturing: Monitoring defect counts on different production lines.
- Environmental Science: Counting occurrences of rare events like earthquakes or rainfall in different regions.

Knowing how to compute joint probabilities, expectations, and variances allows for better risk assessment

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