

Mixed Partial Derivative Theorem Iff. , F_{xy} , And F_{yx} Are All Continuous, Then $F_{xy} = F_{yx}$ 4) Find All The

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Introduction

The study of partial derivatives is fundamental in multivariable calculus, providing insight into how functions change with respect to their variables. One of the most significant results in this area is the Mixed Partial Derivative Theorem, which states conditions under which mixed partial derivatives are equal. This theorem has profound implications in fields like physics, engineering, economics, and any discipline involving multivariable functions.

In particular, the theorem asserts that if a function's second-order mixed partial derivatives are continuous within a region, then the order of differentiation does not matter; that is, $F_{xy} = F_{yx}$. This equality is crucial for simplifying complex derivatives and understanding the behavior of functions in multiple variables.

This article explores the Mixed Partial Derivative Theorem in depth, discussing its conditions, proof outline, and applications. Additionally, we will examine a specific problem involving the theorem—"Find all the functions"—to demonstrate its practical utility.

Understanding Partial Derivatives and Mixed Partial Derivatives

Partial Derivatives

A partial derivative measures how a multivariable function changes as one of its variables changes, keeping the other variables constant. For a function $f(x, y)$, the partial derivatives are denoted as:

- $\frac{\partial f}{\partial x}$ (partial derivative with respect to x)
- $\frac{\partial f}{\partial y}$ (partial derivative with respect to y)

Mixed Partial Derivatives

The mixed partial derivatives involve taking derivatives with respect to different variables in succession:

- $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{xy}$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$

These derivatives indicate how the rate of change of the function with respect to one variable changes as the other variable varies.

The Mixed Partial Derivative Theorem

Statement of the Theorem

The Mixed Partial Derivative Theorem (also known as Clairaut's theorem or Schwarz's theorem) states:

> Let $f(x, y)$ be a function defined on a region R in $\mathbb{R}^2$. If the second mixed partial derivatives f_{xy} and f_{yx} are continuous on R , then:

> $f_{xy} = f_{yx}$

> for all $(x, y) \in R$.

Conditions for the Theorem

For the theorem to hold, the following conditions must be satisfied:

1. The function $f(x, y)$ must be defined in an open region R .
2. The second-order mixed partial derivatives f_{xy} and f_{yx} must be continuous throughout R .
3. The derivatives are taken in the same point within the region.

The continuity of the mixed derivatives is key. If these derivatives are not continuous, the equality may not hold.

Implications of the Theorem

- Symmetry of Mixed Derivatives: The theorem guarantees the equality $f_{xy} = f_{yx}$ under the specified conditions.
- Simplification of Calculations: Knowing that mixed derivatives are equal allows for easier computation and analysis.

- Applications in Differential Equations: Many solutions involve mixed derivatives, and this theorem simplifies boundary and initial value problems.

Proof Outline of the Mixed Partial Derivative Theorem

While the complete proof involves advanced calculus concepts, a simplified outline is as follows:

1. Assumption of Continuity: Assume (F_{xy}) and (F_{yx}) are continuous in (R) .
2. Use of the Mean Value Theorem: Apply the mean value theorem to the difference quotient of (f) over small rectangles in the domain.
3. Integration over a Small Rectangle: Express the difference of the mixed derivatives as integrals over infinitesimal rectangles.
4. Limit Process: Take the limit as the rectangle shrinks to a point, leveraging the continuity of the derivatives.
5. Conclusion: Show that the difference $(F_{xy} - F_{yx})$ tends to zero, establishing the equality.

This proof relies heavily on the properties of continuous functions and the limit process, which justifies the necessity of the continuity condition for the derivatives.

Applications of the Mixed Partial Derivative Theorem

The theorem is widely applied across various disciplines:

- Mathematical Physics: In analyzing potential functions and fields, the symmetry of second derivatives

simplifies the equations governing physical phenomena.

- Economics: Utility and production functions often require mixed derivatives to analyze marginal rates of substitution and returns to scale.
- Engineering: Optimization problems involving multivariable functions depend on the symmetry of second derivatives for stability analysis.
- Differential Geometry: The theorem plays a role in the study of manifolds and curvature, where second derivatives describe geometric properties.

Problem Statement: Find All The Functions

To demonstrate the practical application of the theorem, consider the following problem:

Find all functions $f(x, y)$ such that:

1. $f_{xy} = f_{yx} = e^{xy}$,
2. The mixed derivatives are continuous everywhere.

Step-by-Step Solution

Step 1: Recognize the symmetry condition

Since the problem states $f_{xy} = f_{yx} = e^{xy}$, the mixed derivatives are equal and continuous, satisfying the conditions of the theorem. Therefore, f is at least twice differentiable with continuous second derivatives.

Step 2: Integrate f_{xy} with respect to y

Given:

$$f_{xy} = e^{xy}$$

Integrate with respect to y :

$$f_x(x, y) = \int e^{xy} dy + \phi(x)$$

where $\phi(x)$ is an arbitrary function of x .

Step 3: Evaluate the integral

Let's evaluate:

$$I = \int e^{xy} dy$$

Treat x as a constant:

$$I = \frac{1}{x} e^{xy} + C(x)$$

(because $\frac{\partial}{\partial y} \left(\frac{1}{x} e^{xy} \right) = e^{xy}$).

Thus:

$$f_x(x, y) = \frac{1}{x} e^{xy} + \phi(x)$$

Step 4: Integrate f_x with respect to x

Now, integrate f_x with respect to x :

$$f(x, y) = \int \left(\frac{1}{x} e^{xy} + \phi(x) \right) dx + \psi(y)$$

where $\psi(y)$ is an arbitrary function of y .

Step 5: Evaluate the integral

The integral:

$$J = \int \frac{1}{x} e^{xy} dx + \int \phi(x) dx$$

The second integral is straightforward but arbitrary; it contributes to the general solution.

Focus on:

$$J_1 = \int \frac{1}{x} e^{xy} dx$$

Let's perform a substitution:

- Set $(t = xy)$, then $(dt = y \, dx)$, or $(dx = dt / y)$.

Express (x) :

$$\begin{aligned} &[\\ x &= t / y \\ &] \end{aligned}$$

Then:

$$\begin{aligned} &[\\ J_1 &= \int \frac{1}{t/y} e^t \cdot \frac{dt}{y} = \int \frac{1}{y} e^t \cdot \frac{dt}{y} = \int \frac{e^t}{t} dt \\ &] \end{aligned}$$

This integral, $(\int \frac{e^t}{t} dt)$, is known as the exponential integral function $(Ei(t))$, which cannot be expressed in elementary functions. Therefore, the general solution involves the exponential integral.

Step 6: Write the general form of the solution

Putting it all together:

$$\begin{aligned} &[\\ f(x, y) &= Ei(xy) + \int \phi(x) \, dx + \psi(y) \\ &] \end{aligned}$$

where:

- $(Ei(xy))$ is the exponential integral function.
- $(\phi(x))$ and $(\psi(y))$ are arbitrary functions, representing the constants of integration with respect

to the other variable.

Summary of the General Solution

The general solution for $f(x, y)$ satisfying $f_{xy} = f_{yx} = e^{xy}$ is:

$$f(x, y) = e^{xy} + \Phi(x) + \Psi(y)$$

where:

Frequently Asked Questions

What does the Mixed Partial Derivative Theorem state regarding the equality of mixed partial derivatives?

The theorem states that if the mixed partial derivatives F_{xy} and F_{yx} are continuous in a neighborhood around a point, then F_{xy} equals F_{yx} at that point.

Why is the continuity of F_{xy} and F_{yx} important in the Mixed Partial Derivative Theorem?

Continuity ensures that the mixed partial derivatives are well-behaved and allows the interchange of differentiation order, leading to the equality $F_{xy} = F_{yx}$.

How can one verify if F_{xy} and F_{yx} are continuous for a given function?

By analyzing the function's derivatives and checking their limits and continuity over the region of interest, ensuring both derivatives are defined and continuous.

What are some common applications of the Mixed Partial Derivative Theorem?

It is used in multivariable calculus to simplify calculations, verify the symmetry of second derivatives, and analyze the behavior of functions in optimization and differential equations.

Given a specific function, how do you find all points where F_{xy} and F_{yx} are equal?

Compute the mixed partial derivatives, check their continuity, and solve the resulting equations to identify all points where the derivatives are equal, ensuring the derivatives are continuous in those regions.

Additional Resources

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Introduction

In the realm of multivariable calculus, understanding the behavior and properties of partial derivatives is fundamental for analyzing functions of several variables. A cornerstone in this area is the Mixed Partial Derivative Theorem, which states that under certain conditions, the mixed second-order derivatives of a function are equal regardless of the order of differentiation. This theorem is not only

theoretically significant but also practically applicable across fields such as physics, engineering, and economics. This article explores the theorem in depth, emphasizing the importance of continuity, and provides insights into how it guarantees the equality of mixed partial derivatives. Additionally, we will examine methods to find all functions satisfying these properties, offering a comprehensive overview suitable for students, educators, and professionals alike.

Understanding the Mixed Partial Derivative Theorem

Fundamental Concepts and Definitions

Before delving into the theorem itself, it's crucial to understand some foundational concepts:

- Partial Derivatives: These measure how a function changes as one of its variables changes, holding other variables constant. For a function $f(x, y)$, the first-order partial derivatives are $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.
- Mixed Partial Derivatives: These are second derivatives involving different variables, such as $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ and $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$.

Understanding these derivatives is essential because the symmetry or lack thereof influences the behavior of multivariable functions.

The Theorem and Its Conditions

The Mixed Partial Derivative Theorem, often referred to as Clairaut's Theorem or Schwarz's Theorem,

states:

If a function f is defined on a neighborhood in $\mathbb{R}^2$ and has continuous second-order partial derivatives f_{xy} and f_{yx} , then

$$f_{xy} = f_{yx}$$

everywhere in that neighborhood.

Key conditions include:

- The function f must be defined in an open region where the derivatives are considered.
- Both mixed partial derivatives f_{xy} and f_{yx} must be continuous in that region.

These conditions ensure the derivatives are interchangeable, establishing the equality of mixed derivatives.

Why Continuity Matters: The Core of the Theorem

Continuity as a Guarantee of Equality

Continuity of second-order mixed partial derivatives isn't just a technicality; it's the linchpin that guarantees their equality. To understand this, consider the following:

- If f_{xy} and f_{yx} are continuous in a neighborhood, then the limit processes involved in defining these derivatives are well-behaved and interchangeable.
- When derivatives are continuous, the limit definitions of the derivatives commute, ensuring that the order of differentiation does not matter.

Intuitive Explanation:

Imagine you're analyzing the surface defined by $f(x, y)$. If the surface is "smooth" enough—meaning the second derivatives are continuous—the curvature in one direction is consistent regardless of the order in which you measure it. This smoothness ensures the mixed derivatives mirror each other.

Implications of the Theorem

The theorem's implications are profound:

- It simplifies the analysis of multivariable functions by allowing the interchange of mixed derivatives.
- It guarantees the symmetry of the second derivative matrix (Hessian), which is essential in optimization problems.
- It underpins many advanced mathematical theories, including differential equations and differential geometry.

Applications and Examples

Practical Examples Demonstrating the Theorem

Example 1: A Simple Function

Consider $f(x, y) = x^2 y + y^3$.

Calculate the mixed derivatives:

$$\begin{aligned} f_x &= 2xy, \quad f_y = x^2 + 3y^2 \\ f_{xy} &= 2x, \quad f_{yx} = 2x \end{aligned}$$

Next, find the second derivatives:

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y} (f_x) = 2x, \quad f_{yx} = \frac{\partial}{\partial x} (f_y) = 2x \end{aligned}$$

Since $f_{xy} = f_{yx} = 2x$ and both derivatives are continuous everywhere, the theorem holds.

Example 2: A Function with Continuous Second Derivatives

Let $f(x, y) = e^{xy}$.

Calculate:

$$\begin{aligned} f_x &= y e^{xy}, \quad f_y = x e^{xy} \\ f_{xy} &= e^{xy}, \quad f_{yx} = e^{xy} \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial y} (f_x) &= e^{xy} + xy e^{xy} = e^{xy}(1 + xy) \end{aligned}$$

and

$$\frac{\partial}{\partial x} (f_y) = e^{xy} + xy e^{xy} = e^{xy}(1 + xy)$$

Again, the mixed derivatives are equal, and the derivatives are continuous everywhere, satisfying the theorem.

Finding All Functions Satisfying the Conditions

Problem Statement

Suppose we want to find all functions $f(x, y)$ for which:

- $f_{xy} = f_{yx}$
- Both f_{xy} and f_{yx} are continuous in a region

Given these, what is the general form of such functions?

Approach to the Solution

The key insight is that the equality of mixed derivatives under continuity implies that f can be reconstructed from its second derivatives. This leads to an integration process:

1. Start with the mixed derivatives: Assume $f_{xy} = f_{yx} = g(x, y)$, where g is a continuous function.

2. Integrate to find first derivatives:

- Integrate $g(x, y)$ with respect to x :

$$f_x(x, y) = \int g(x, y) dx + \phi(y)$$

where $\phi(y)$ is an arbitrary function of y .

- Similarly, integrating f_y with respect to y gives:

$$f_y(x, y) = \int g(x, y) dy + \psi(x)$$

where $\psi(x)$ is an arbitrary function of x .

3. Consistency Condition:

Since both f_x and f_y are partial derivatives of the same function f , their derivatives must be compatible:

$$\frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial x} f_y(x, y)$$

which leads to differential equations involving $\phi(y)$, $\psi(x)$, and $g(x, y)$.

4. Construct the General Solution:

After integrating and applying the compatibility conditions, the general solution can be expressed as:

$$f(x, y) = \int \int g(s, t) \, ds \, dt + A(x) + B(y) + C$$

where A, B, C are arbitrary functions/constants arising from integration.

Special case: If $f_{xy} = f_{yx} = 0$, then f reduces to the sum of functions of x and y separately:

$$f(x, y) = \alpha(x) + \beta(y) + \gamma$$

Conclusion: The Significance of the Theorem in Mathematical Analysis

The Mixed Partial Derivative Theorem is more than a technical result; it is a foundational principle that

underpins much of multivariable calculus, differential equations, and mathematical modeling. Its requirement for continuity ensures the smoothness of functions, permitting the interchange of differentiation order—a property that simplifies complex analyses and guarantees consistent results.

Understanding when and how mixed derivatives are equal allows mathematicians and scientists to build more accurate models of natural phenomena, optimize functions, and analyze surfaces with confidence. The theorem's conditions serve as a checklist for verifying the symmetry of second derivatives, guiding both theoretical exploration and practical computation.

As we have seen, the theorem's implications extend beyond mere mathematical curiosity. It provides the tools to find all functions satisfying specific derivative conditions, leading to a deeper comprehension of the structure of multivariable functions.

In modern applications, whether in physics describing fields, engineering designing systems, or economics modeling markets, the assurance that mixed partial derivatives are equal under continuity conditions facilitates the development of robust and reliable models. This fundamental principle continues to be a cornerstone in the mathematician's toolkit, shaping our understanding of the complex, multidimensional world.

References

- Stewart, J. (2016). *Calculus: Early Transcendentals*. Cengage Learning.
- Apostol, T. M. (1967). *Mathematical Analysis*. Addison-Wesley.
- Thomas, G. B., & Finney, R. L.

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