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Understanding how to compute the area of a region bounded by curves is a fundamental topic in calculus, particularly when dealing with definite integrals. When faced with a problem asking to find a definite integral whose value equals the area of a specific region, it's essential to understand the geometric interpretation of integrals and the techniques involved in calculating such areas. This article provides a comprehensive guide to finding and evaluating definite integrals that represent the area of regions bounded by various curves.

Introduction to Area Calculation Using Definite Integrals

The core principle behind using definite integrals to find areas is based on the concept of summing infinitesimal slices of a region. When a region is bounded by curves, the area can often be expressed as the integral of a function over a specific interval.

Key Concept:

- The definite integral $\int_a^b f(x) \, dx$ calculates the net area between the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.
- If the function $f(x)$ is positive on $[a, b]$, then the definite integral directly gives the area of the region.
- If $f(x)$ is negative on the interval, the integral gives the signed area, which can be negative. To find the actual area, take the absolute value of the function or split the integral at points where the function crosses the x-axis.

Why is this important?

In problems asking to find a definite integral whose value equals the area of a bounded region, the key is to identify the correct limits and the appropriate function or combination of functions that define the boundary.

Understanding Regions Bounded by Curves

The region bounded by curves can take many forms: a simple area under a single curve, a region between two curves, or a more complex shape involving multiple boundary lines.

Types of bounded regions:

1. **Region under a single curve**: e.g., between $x=a$ and $x=b$, bounded above by $y=f(x)$.
2. **Region between two curves**: e.g., between $y=f(x)$ and $y=g(x)$, over an interval $[a, b]$.
3. **Region bounded by curves and lines**: including vertical or horizontal lines forming closed shapes.

Example:

Suppose you have the curves $y = x^2$ and $y = 4 - x$. The bounded region is enclosed where these two curves intersect, and calculating its area involves integrating the difference between the functions over the intersection interval.

Step-by-Step Approach to Finding the Definite Integral for the Area

When asked to find a definite integral representing the area of a bounded region, follow these steps:

1. Sketch the Region

- Draw the involved curves and identify the region.
- Mark points of intersection to determine limits of integration.
- Note whether the region lies above or below the x-axis.

2. Find Intersection Points

- Solve for where the boundary curves intersect to find the limits a and b.
- These points set the bounds of the integral.

3. Determine the Expression for the Area

- For a region between $y = f(x)$ and $y = g(x)$, the area is given by:

$$
\text{Area} = \int_a^b |f(x) - g(x)| \, dx
$$

- If $f(x) \geq g(x)$ over $[a, b]$, then:

$$
\text{Area} = \int_a^b (f(x) - g(x)) \, dx
$$

4. Set Up the Integral

- Write the integral with the correct limits and integrand.
- Ensure the integrand reflects the boundary functions properly.

5. Evaluate the Integral

- Use analytical methods (antiderivatives) to compute the integral.
- For complex functions, consider substitution or numerical approximation techniques.

6. Confirm the Result

- Verify that the computed integral matches the geometric area.
- If necessary, adjust the integral limits or integrand based on the region's shape.

Examples of Finding a Definite Integral for a Bounded Region

Let’s explore some concrete examples to solidify understanding.

Example 1: Area Under a Single Curve

Suppose the problem states:

> Find a definite integral whose value is the area of the region bounded by the curve $y = x^2$ between $x=0$ and $x=2$.

Solution:

- The region is the area under $y = x^2$ from 0 to 2.
- Since $y = x^2 \geq 0$ in this interval, the area is:

$$
\boxed{
\int_0^2 x^2 \, dx
}
$$

- Evaluating:

$$
\int_0^2 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{2^3}{3} - 0 = \frac{8}{3}
$$

Thus, the definite integral $\int_0^2 x^2 \, dx$ equals the area of the region.

Example 2: Region Between Two Curves

Suppose you need to find a definite integral that equals the area between $y = x$ and $y = x^2$ over the interval where they intersect.

- Find intersection points:

$$
x = x^2 \implies x^2 - x = 0 \implies x(x - 1) = 0 \implies x=0, x=1
$$

- The region is between $x=0$ and $x=1$, bounded by the curves.

- Since $y = x$ is above $y = x^2$ on $[0,1]$, the area is:

$$
\text{Area} = \int_0^1 (x - x^2) \, dx
$$

- Evaluating:

$$
\int_0^1 (x - x^2) \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - 0 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}
$$

Hence, $\int_0^1 (x - x^2) \, dx = \frac{1}{6}$ represents the area.

Advanced Techniques for Complex Regions

Some bounded regions require more advanced methods, especially when the boundary curves are complicated or involve multiple parts.

Techniques include:

- **Splitting the integral**: When the region crosses the x-axis, split the integral at points where the function changes sign.
- **Using symmetry**: Exploit symmetry properties to simplify calculations.

- **Changing variables or substitution**: For complex functions, substitution can make integration manageable.
- **Polar coordinates**: For regions with circular symmetry, convert to polar coordinates to simplify the integral.

Example of splitting an integral:

Suppose the region between $y = \sin x$ and $y=0$ over $[0, 2\pi]$. Since $\sin x$ is positive between 0 and π, and negative between π and 2π, the total area is:

$$
\text{Area} = \int_0^{\pi} \sin x \, dx - \int_\pi^{2\pi} \sin x \, dx
$$

Calculating:

$$
\int_0^{\pi} \sin x \, dx = 2
$$
$$
\int_\pi^{2\pi} \sin x \, dx = 0 - (-2) = 2
$$

Total area:

$$
2 + 2 = 4
$$

Note: Since the integral of $\sin x$ over $[\pi, 2\pi]$ is negative, taking the absolute value or splitting at zeros ensures the area is positive.

Applications of Finding Areas via Definite Integrals

Calculating areas with definite integrals is not just an academic exercise but has real-world applications, including:

- **Physics**: Computing work done by a force, center of mass, or electric charge distributions.