Prove That There Exists A Linear Transformation L: R2 R3 Such That L(1, 1) = (1,0,2) And L(2,3)= (1,-1,

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Introduction

In the realm of linear algebra, understanding the existence and construction of linear transformations between vector spaces is fundamental. This article aims to demonstrate the existence of a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^3$ that satisfies specific conditions: $L(1,1) = (1,0,2)$ and $L(2,3) = (1,-1, \ast \ast)$. While the second image seems incomplete, we will interpret the task as proving the general approach to such problems, focusing on the conditions for the existence of a linear transformation given certain images of vectors.

Understanding Linear Transformations

A linear transformation $L: V \to W$ between vector spaces V and W over the same field (here, real numbers $\mathbb{R}$) must satisfy two properties:

1. Additivity: $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$.

2. Homogeneity: $L(c \mathbf{v}) = c L(\mathbf{v})$ for all $c \in \mathbb{R}$, $\mathbf{v} \in V$.

Given these properties, linear transformations are completely determined by their action on a basis of the domain space. Therefore, to define L, it suffices to specify its images on a basis of $\mathbb{R}^2$.

Constructing the Transformation: Basis and Conditions

Since $\mathbb{R}^2$ is a 2-dimensional vector space, a natural basis is:

$$
\mathcal{B} = \{ (1, 0), (0, 1) \}
$$

Any vector $(x, y) \in \mathbb{R}^2$ can be written as:

$$
(x, y) = x (1, 0) + y (0, 1)
$$

Thus, to define a linear transformation L, we need to specify:

$$
L(1, 0) = \mathbf{v}_1 \in \mathbb{R}^3
$$
$$
L(0, 1) = \mathbf{v}_2 \in \mathbb{R}^3
$$

Once $\mathbf{v}_1$ and $\mathbf{v}_2$ are known, the transformation L acts as:

$$
L(x, y) = x \mathbf{v}_1 + y \mathbf{v}_2
$$

Key Point: To satisfy $L(1,1) = (1,0,2)$ and $L(2,3) = (1, -1, \ast \ast)$, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ must meet the following conditions:

$$
L(1,1) = \mathbf{v}_1 + \mathbf{v}_2 = (1, 0, 2)
$$
$$
L(2,3) = 2 \mathbf{v}_1 + 3 \mathbf{v}_2 = (1, -1, \ast \ast)
$$

Determining the Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

Formulating the System of Equations

Set:

$$
\mathbf{v}_1 = (a_1, a_2, a_3)
$$
$$
\mathbf{v}_2 = (b_1, b_2, b_3)
$$

From the conditions:

$$
\mathbf{v}_1 + \mathbf{v}_2 = (1, 0, 2)
$$
$$
2 \mathbf{v}_1 + 3 \mathbf{v}_2 = (1, -1, c)
$$

where c is the unknown third component in the second image, which we denote as c.

This gives us a system of six equations:

1. $a_1 + b_1 = 1$
2. $a_2 + b_2 = 0$
3. $a_3 + b_3 = 2$

4. $2a_1 + 3b_1 = 1$
5. $2a_2 + 3b_2 = -1$
6. $2a_3 + 3b_3 = c$

Solving the System

From equations (1) and (4):

$$
a_1 + b_1 = 1
$$
$$
2a_1 + 3b_1 = 1
$$

Express a_1 from (1):

$$
a_1 = 1 - b_1
$$

Substitute into (4):

$$
2(1 - b_1) + 3b_1 = 1
$$
$$
2 - 2b_1 + 3b_1 = 1
$$
$$
2 + b_1 = 1
$$
\[

b_1 = -1
\]

Then:

\[
a_1 = 1 - (-1) = 2
\]

Similarly, for the second component:

\[
a_2 + b_2 = 0
\]
\[
2a_2 + 3b_2 = -1
\]

Express \(a_2 = -b_2 \). Substituting into the second:

\[
2(-b_2) + 3b_2 = -1
\]
\[
-2b_2 + 3b_2 = -1
\]
\[
b_2 = -1
\]
\[
a_2 = -b_2 = 1
\]

For the third component:

\[
a_3 + b_3 = 2
\]
\[
2a_3 + 3b_3 = c
\]

Express \(a_3 = 2 - b_3 \). Substituting into the second:

\[
2(2 - b_3) + 3b_3 = c
\]
\[
4 - 2b_3 + 3b_3 = c
\]
\[

$$
4 + b_3 = c
$$

Thus, $c = 4 + b_3$. Since b_3 is arbitrary, the third component of the image in the second transformation is not fixed by the given data, implying that:

$$
\boxed{
\mathbf{v}_1 = (2, 1, a_3), \quad \mathbf{v}_2 = (-1, -1, b_3), \quad c = 4 + b_3
}
$$

where $a_3, b_3 \in \mathbb{R}$ are arbitrary parameters.

Existence of the Linear Transformation

Key Observations

- The above solution shows that for any choice of a_3 and b_3, we can define vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that satisfy the given images for $(1,1)$ and $(2,3)$.
- Since these vectors form the images of the basis vectors $(1, 0)$ and $(0, 1)$, the linear transformation L is uniquely determined by:

$$
L(x, y) = x \mathbf{v}_1 + y \mathbf{v}_2
$$

- The only restriction arises from the third component of $L(2,3)$, which depends on the parameter b_3.

Conditions for the Transformation to Exist

- The key requirement is that the images $L(1,1)$ and $L(2,3)$ must be consistent with linearity.
- Our calculations demonstrate that these images are compatible, provided the third component of $L(2,3)$ is chosen as $c = 4 + b_3$.

Therefore, a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^3$ exists that maps $(1,1)$ to $(1,0,2)$ and $(2,3)$ to $(1, -1, c)$ for any c satisfying $c = 4 + b_3$.

Explicit Construction

Frequently Asked Questions

How can we determine if there exists a linear transformation L: R2 → R3 such that L(1,1) = (1,0,2) and L(2,3) = (1,-1,?)?

To determine this, we need to check if the images of the vectors are consistent with linearity. Specifically, verify if L(2,3) can be expressed as 2·L(1,1) + 1·L(0,1). Since L(0,1) is unknown, we check if the given images satisfy the linearity conditions to confirm the existence of such a transformation.

What conditions must the vectors (1,1) and (2,3) satisfy to ensure a linear transformation L: R2 → R3 exists with specified images?

They must be linearly independent or compatible in the sense that the images under L should satisfy linearity. Specifically, the image of (2,3) should be equal to 2·L(1,1) + 3·L(0,1), assuming L(0,1) exists. Consistency of these relations confirms the existence of L.

Given L(1,1) = (1,0,2), what additional information is needed to define L completely?

We need to know L(0,1) to fully define L, since any vector in R2 can be expressed as a linear combination of (1,1) and (0,1). With L(0,1), we can determine L(x,y) for any (x,y).

Is it necessary for the images of basis vectors to be linearly independent for the existence of a linear transformation L with specified images?

No, linear independence of the images is not strictly necessary for existence, but it is necessary for L to be injective. For existence, the images must be consistent with linearity; linear independence affects invertibility but not existence.

How do we verify if the given images satisfy the properties of a linear transformation?

We verify if the images satisfy the additive and scalar multiplication properties, i.e., L(u+v) = L(u) + L(v) and L(cu) = cL(u). In the context of given points, check if the images are consistent with these properties for linear combinations of vectors.

Can we construct a linear transformation L: R2 → R3 given only L(1,1) and L(2,3)?

Not completely, unless the images of these vectors are

compatible with linearity and the images of any other basis vectors are determined accordingly. Typically, knowing images of basis vectors suffices to define L uniquely.

What is the significance of the image L(2,3) in relation to L(1,1) in determining the existence of L?

If L(2,3) equals 2·L(1,1) + 1·L(0,1), then the images are consistent with linearity. If not, such a linear transformation cannot exist with the specified images unless additional vectors are specified.

How do you prove that a linear transformation L exists with the given conditions?

Construct L by defining its action on basis vectors, ensuring the images are consistent with the given data. If the images satisfy the linearity conditions, then L exists. In this case, you check if L(2,3) = 2·L(1,1) + 3·L(0,1), which requires knowing L(0,1). If consistent, the transformation exists.

Additional Resources

Prove That There Exists A Linear Transformation L: $R^2 \rightarrow R^3$ Such That L(1, 1) = (1, 0, 2) And L(2, 3) = (1, -1, ?)

When working with linear algebra, a common problem involves determining whether a linear transformation with specified properties exists. Specifically, for the problem Prove That There Exists A Linear Transformation L: $R^2 \rightarrow R^3$ Such That L(1, 1) = (1, 0, 2) And L(2, 3) = (1, -1, ?), we need to understand the conditions under which such a transformation can exist, and how to explicitly construct it if possible. This article will guide you through the process step-by-step, exploring the fundamental concepts involved, and ultimately demonstrating whether such a transformation exists and how to find it.

Understanding the Problem

Before diving into the proof, let's clarify what is being asked:

- Given: Two vectors in R^2, namely (1, 1) and (2, 3), and their images under some linear transformation L, which are specified as (1, 0, 2) and (1, -1, ?), respectively.
- Question: Does there exist a linear transformation L: $R^2 \rightarrow R^3$ satisfying these conditions? If yes, how can we

explicitly construct it? If not, why?

Note that the second image contains an unknown "?". Usually, in such problems, the unknown component can be determined from the linearity condition, provided the images are consistent.

Linear Transformations and Their Properties

What is a linear transformation?

A linear transformation L from R^2 to R^3 is a function that satisfies two properties for all vectors u, v $\in R^2$ and all scalars c:

1. Additivity: L(u + v) = L(u) + L(v)
2. Homogeneity: L(cu) = cL(u)

Any linear transformation from R^2 to R^3 can be represented as a matrix multiplication:

$$L(\mathbf{x}) = A \mathbf{x}$$

where A is a 3×2 matrix, and $\mathbf{x}$ is a vector in R^2.

Step 1: Expressing the Transformation as a Matrix

Suppose the matrix representation of L is:

$$
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix}
$$

Given a vector $\mathbf{x} = (x_1, x_2)$, the transformation is:

$$
L(\mathbf{x}) = A \mathbf{x} = \begin{bmatrix}
a_{11}x_1 + a_{12}x_2 \\
a_{21}x_1 + a_{22}x_2 \\
a_{31}x_1 + a_{32}x_2
\end{bmatrix}
$$

Our goal is to find the matrix A satisfying the given conditions.

Step 2: Applying the Given Conditions

From the problem:

$$
L(1, 1) = (1, 0, 2)
$$

$$
L(2, 3) = (1, -1, ?)
$$

Expressed in terms of A:

$$
A \begin{bmatrix}1 \\ 1 \end{bmatrix} = \begin{bmatrix}1 \\ 0 \\ 2 \end{bmatrix}
$$

$$
A \begin{bmatrix}2 \\ 3 \end{bmatrix} = \begin{bmatrix}1 \\ -1 \\ ? \end{bmatrix}
$$

which translates into systems of equations for the entries of A.

Step 3: Deriving the Equations

Using the matrix form:

$$
A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 \\
a_{21}x_1 + a_{22}x_2 \\
a_{31}x_1 + a_{32}x_2
\end{bmatrix}
$$

Applying this to the vectors:

- For (1, 1):

$$
\begin{cases}
a_{11} \times 1 + a_{12} \times 1 = 1 \\
a_{21} \times 1 + a_{22} \times 1 = 0 \\
a_{31} \times 1 + a_{32} \times 1 = 2
\end{cases}
$$

- For (2, 3):

$$
\begin{cases}
a_{11} \times 2 + a_{12} \times 3 = 1 \\
a_{21} \times 2 + a_{22} \times 3 = -1 \\
a_{31} \times 2 + a_{32} \times 3 = ?
\end{cases}
$$

\]

Step 4: Solving for the Matrix Entries

From the first set:

\[
\begin{cases}
a_{11} + a_{12} = 1 \quad (1) \\
a_{21} + a_{22} = 0 \quad (2) \\
a_{31} + a_{32} = 2 \quad (3)
\end{cases}
\]

From the second set:

\[
\begin{cases}
2a_{11} + 3a_{12} = 1 \quad (4) \\
2a_{21} + 3a_{22} = -1 \quad (5) \\
2a_{31} + 3a_{32} = ? \quad (6)
\]

Step 5: Find the Unknowns

Solving for a_{11} and a_{12}:

From (1):

$$
a_{11} = 1 - a_{12}
$$

Substitute into (4):

$$
2(1 - a_{12}) + 3a_{12} = 1
$$
$$
2 - 2a_{12} + 3a_{12} = 1
$$
$$
2 + a_{12} = 1
$$
$$
a_{12} = -1
$$

Then,

$$
a_{11} = 1 - (-1) = 2
$$

Solving for a_{21} and a_{22}:

From (2):

$$
a_{21} = -a_{22}
$$

Substitute into (5):

$$
2(-a_{22}) + 3a_{22} = -1
$$

$$
-2a_{22} + 3a_{22} = -1
$$

$$
a_{22} = -1
$$

then,

$$
a_{21} = -(-1) = 1
$$

Solving for a_{31} and a_{32}:

From (3):

$$
a_{31} = 2 - a_{32}
$$

Substitute into (6):

$$
2(2 - a_{32}) + 3a_{32} = ?
$$

$$
4 - 2a_{32} + 3a_{32} = ?
$$

$$
4 + a_{32} = ?
$$

Note that the value of a_{32} is not constrained by the previous equations. Since the question mark in the image is unspecified, this indicates that the third coordinate's second component is not fixed by the initial data.

Conclusion:

Existence of the Linear Transformation:

- The entries of the matrix A are:

$$
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix} = \begin{bmatrix}
2 & -1 \\
1 & -1 \\
2 - a_{32} & a_{32}
\end{bmatrix}
$$

- The only remaining degree of freedom is a_{32}, which is arbitrary. The value of the third component of $L(2, 3)$ is:

$$
a_{31} \times 2 + a_{32} \times 3 = (2 - a_{32}) \times 2 + 3a_{32} = 4 - 2a_{32} + 3a_{32} = 4 + a_{32}
$$

- Since a_{32} can be any real number, the third coordinate of $L(2, 3)$ is not uniquely determined, but it exists for any choice of a_{32}.

Thus, the linear transformation L exists, and it can be explicitly constructed for any value of a_{32}. The general form of such a transformation is:

\[
L(x, y) = \begin{bmatrix}
**2 & -1 **
**1 & -1 **
2 - a_{32} & a_{32}
****

[Prove That There Exists A Linear Transformation L R2 R3 Such That L 1 1 1 0 2 And L 2 3 1 1]

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