Rewrite The Following Integral Using The Indicated Order Of Integration And Then Evaluate The Resulting

Rewrite The Following Integral Using The Indicated Order Of Integration And Then Evaluate The Resulting problem is a fundamental concept in calculus, especially in the context of multiple integrals. Changing the order of integration can simplify the process of evaluation significantly. Many students and practitioners often encounter complex integrals where the order of integration can be rearranged to make the problem more manageable. In this article, we will explore the techniques involved in rewriting integrals with different orders of integration, illustrate the process with detailed examples, and demonstrate how to evaluate the resulting integrals efficiently.

Understanding Double Integrals and Their Orders of Integration

What Are Double Integrals?

Double integrals are used to compute the volume under a surface over a specified region in the xy-plane. They are expressed in the form:

$$
\iint_{R} f(x,y) \, dA
$$

where R is the region of integration in the xy-plane, and $f(x,y)$ is the integrand function.

Why Change the Order of Integration?

Changing the order of integration can be advantageous for several reasons:

- Simplification of the integrand: Certain orders may lead to easier integrals.
- Easier determination of limits: The limits of integration might be more straightforward in one order.
- Handling complex regions: Some regions are naturally described with one order but complicated in another.

Basic Principles for Reordering

Before changing the order, it is crucial to understand the region R. The typical approach involves:

1. Sketching the region: To visualize the limits.
2. Describing the region in terms of inequalities: To identify the bounds for x and y.
3. Expressing the integral in the new order: By swapping the roles of x and y.

Step-by-Step Approach to Rewriting and Evaluating Integrals

Step 1: Analyze the Region of Integration

Start with the original integral and carefully examine the bounds. For example, an integral might be expressed as:

$$
\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx
$$

or

$$
\int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy
$$

Step 2: Sketch the Region R

Plotting the region helps visualize the limits and understand how to switch the order of integration.

Step 3: Determine the New Limits

Identify the corresponding bounds for the new order. This might involve solving inequalities to express x in terms of y or vice versa.

Step 4: Write the Reordered Integral

Express the original integral with the new limits, ensuring the order of integration now aligns with the new description of the region.

Step 5: Evaluate the Integral

Perform the integration step-by-step, often starting with the inner integral, followed by the outer one.

Practical Example: Rewriting and Evaluating an Integral

Let's consider a concrete example to illustrate the process.

Given Integral

Evaluate:

$$
I = \int_{x=0}^{1} \int_{y=0}^{x^2} (x + y) \, dy \, dx
$$

Step 1: Analyze the Region

The bounds indicate:

- x varies from 0 to 1.
- For each fixed x, y varies from 0 up to x^2.

Step 2: Sketch the Region

Plotting:

- The region is bounded below by $y=0$.
- The upper boundary is $y = x^2$.
- x ranges from 0 to 1.

This describes the region under the parabola $y = x^2$ between $x=0$ and $x=1$.

Step 3: Find the Equivalent Limits in Terms of y

To switch the order:

- For a fixed y, determine the range of x.

From $y \leq x^2$, we have:

$$
x \geq \sqrt{y}
$$

Since x varies from 0 to 1, and y varies from 0 up to $y = 1^2 = 1$, the limits are:

- y from 0 to 1.
- For each y, x from $\sqrt{y}$ to 1.

Step 4: Write the Reordered Integral

The integral becomes:

$$
I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} (x + y) \, dx \, dy
$$

Step 5: Evaluate the Inner Integral

Compute:

$$
\int_{\sqrt{y}}^{1} (x + y) \, dx
$$

which splits into:

$$
\int_{\sqrt{y}}^{1} x \, dx + y \int_{\sqrt{y}}^{1} dx
$$

Calculate each:

- $\int x \, dx = \frac{x^2}{2}$

- $\int dx = x$

So:

$$
\left[\frac{x^{2}}{2}\right]_{\sqrt{y}}^{1} + y [x]_{\sqrt{y}}^{1} = \left(\frac{1^2}{2} - \frac{(\sqrt{y})^2}{2}\right) + y (1 - \sqrt{y})
$$

Simplify:

$$
\left(\frac{1}{2} - \frac{y}{2}\right) + y(1 - \sqrt{y}) = \frac{1 - y}{2} + y - y \sqrt{y}
$$

Now, the entire integral becomes:

$$
I = \int_{0}^{1} \left(\frac{1 - y}{2} + y - y \sqrt{y} \right) dy
$$

which simplifies to:

$$
I = \int_{0}^{1} \left(\frac{1 - y}{2} + y - y^{3/2} \right) dy
$$

Step 6: Final Integration

Break into parts:

$$
I = \int_{0}^{1} \frac{1 - y}{2} dy + \int_{0}^{1} y \, dy - \int_{0}^{1} y^{3/2} dy
$$

Compute each:

1. $\int_{0}^{1} \frac{1 - y}{2} dy = \frac{1}{2} \int_{0}^{1} (1 - y) dy = \frac{1}{2} \left[y - \frac{y^2}{2} \right]_0^1 = \frac{1}{2} \left(1 - \frac{1}{2} \right) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

2. $\int_{0}^{1} y \, dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}$

3. $\int_{0}^{1} y^{3/2} dy = \left[\frac{y^{5/2}}{\frac{5}{2}} \right]_0^1 = \frac{2}{5} y^{5/2} \big|_0^1 = \frac{2}{5} \times 1 = \frac{2}{5}$

Adding up:

$$
I = \frac{1}{4} + \frac{1}{2} - \frac{2}{5} = \frac{1}{4} + \frac{2}{4} - \frac{2}{5}
$$

Express all with common denominator 20:

$$
\frac{5}{20} + \frac{10}{20} - \frac{8}{20} = \frac{5 + 10 - 8}{20} = \frac{7}{20}
$$

Final Result:

$$
\boxed{\frac{7}{20}}
$$

Additional Tips for Reordering Integrals

- Always sketch the region first to avoid errors.
- Convert inequalities carefully.
- Verify the new limits by testing points within the region.
- Be mindful of the order of the limits; they must correctly correspond to the region boundaries.
- When integrating, simplify the integrand as much as possible before integration.

Common Mistakes to Avoid

- Misinterpreting the region and setting incorrect limits.
- Forgetting to switch the bounds appropriately when changing the order.
- Overlooking the domain restrictions of the functions involved.
- Failing to simplify the integrand after rewriting.

Conclusion

Rewriting an integral using a different order of integration is a powerful technique in multivariable calculus that can greatly simplify the evaluation process. By carefully analyzing the region, sketching it, and determining the correct bounds, you can transform complex integrals into more manageable forms. Practice with various examples enhances intuition and proficiency, enabling you to approach double integrals confidently. Remember, the key steps involve understanding

Frequently Asked Questions

What does it mean to rewrite an integral using the indicated order of integration?

Rewriting an integral using the indicated order of integration involves changing the order in which the integration variables are processed, often by analyzing the region of integration and expressing it in a different but equivalent way to simplify evaluation.

Why is changing the order of integration useful when evaluating double integrals?

Changing the order of integration can simplify the integral, make the limits easier to evaluate, or enable the use of different integration techniques, thereby making the calculation more straightforward.

How do you determine the new limits of integration when rewriting an integral in a different order?

You analyze the region of integration by sketching or describing it, then express the inequalities defining the region in terms of the other variable order, which provides the new limits for the rewritten integral.

Can you provide a general method for rewriting a double integral in terms of different order of integration?

Yes. First, identify the region of integration, sketch it if necessary, then solve the inequalities defining the region for the other variable, and finally write the integral with the new limits reflecting this reordering.

What are common mistakes to avoid when rewriting integrals with different orders of integration?

Common mistakes include misidentifying the region of integration, incorrectly determining the new limits, and overlooking the bounds of the variables, which can lead to incorrect results.

How do you evaluate the resulting integral after rewriting it in a different order?

Once the integral is rewritten with the new limits, evaluate it step-by-step, performing the inner integral first, then the outer integral, using standard integration techniques.

Is it always advantageous to change the order of

integration?

Not always. While it can simplify the integral, sometimes the original order is easier or more straightforward. The decision depends on the specific integral and the region of integration.

What tools or techniques can help in rewriting integrals in different orders?

Graphing the region, setting up inequalities, and using substitution or parametric descriptions can assist in understanding and rewriting the limits for different orders of integration.

Can you give an example of rewriting an integral from dxdy to dy dx order?

For example, if the region is defined by $0 \leq y \leq 1$ and $0 \leq x \leq y$, rewriting the integral from dxdy to dy dx involves expressing x in terms of y, resulting in $0 \leq y \leq 1$ and $0 \leq x \leq y$.

After rewriting the integral, how do you verify that the limits are correct?

You verify by graphing the region or testing sample points within the limits to ensure they correspond to the original region, confirming the rewritten limits correctly represent the same area.

Additional Resources

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Introduction

In the realm of multivariable calculus, the process of evaluating double integrals often involves strategic manipulation of the order of integration. This technique not only simplifies complex integrals but also provides deeper insights into the geometric and algebraic structure of the problem. The purpose of this article is to thoroughly investigate the process of rewriting a given double integral using a different order of integration, followed by its evaluation. Through meticulous analysis and step-by-step methodology, we aim to elucidate the principles, strategies, and common pitfalls associated with this essential calculus skill.

Theoretical Background

Understanding Double Integrals

A double integral extends the concept of integration to functions of two variables, typically over a region R in the xy-plane:

$$
\iint_{R} f(x, y) \, dA
$$

where dA can be expressed as $dx\,dy$ or $dy\,dx$, depending on the order of integration.

Order of Integration: Significance and Flexibility

The order of integration refers to whether we integrate with respect to x first or y first. Changing the order can:

- Simplify the integral if one order leads to easier limits or integrand.
- Make the integral more amenable to standard techniques or known integrals.
- Reveal the geometric nature of the region R.

Fubini’s Theorem

Fubini’s Theorem states that if f is continuous on a rectangular region, then the double integral can be computed as an iterated integral in either order:

$$
\iint_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy\, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx\, dy
$$

This theorem underpins the flexibility in changing the order of integration.

Case Study: A Specific Integral

Suppose we are given the integral:

$$
I = \int_{0}^{1} \int_{x}^{1} e^{y^2} \, dy\, dx
$$

Our goal is:

1. To rewrite this integral with the order of integration reversed.
2. To evaluate the integral after the change.

Step 1: Analyzing the Region of Integration

Visualizing the Region R

The current limits describe the region:

- x varies from 0 to 1.
- For each fixed x, y varies from $y = x$ to $y = 1$.

Graphically, this corresponds to the region:

$$
R = \{ (x, y) \mid 0 \le x \le 1, \, x \le y \le 1 \}
$$

This is a triangular region bounded by:

- $y = x$,
- $y = 1$,
- $x = 0$.

Expressing the Region in y-First Limits

To switch the order, we need to describe R in terms of y:

- For a fixed y, the x-limits are determined by the inequalities:

$$
0 \le x \le y
$$

- Since y varies from 0 to 1 (the lowest y on the region is 0, the highest is 1), the y-limits are:

$$
0 \le y \le 1
$$

Thus, the rewritten limits are:

$$
x \text{ from } 0 \text{ to } y, \quad y \text{ from } 0 \text{ to } 1
$$

Step 2: Rewriting the Integral

Using the new limits, the integral becomes:

\[

I = \int_{0}^{1} \int_{0}^{y} e^{y^2} \, dx \, dy
\]

Note that the integrand \(e^{y^2} \) is independent of \(x \), so integrating with respect to \(x \) is straightforward:

\[
I = \int_{0}^{1} \left[\int_{0}^{y} e^{y^2} \, dx \right] dy = \int_{0}^{1} e^{y^2} \left(\int_{0}^{y} dx \right) dy
\]

\[
I = \int_{0}^{1} e^{y^2} \cdot y \, dy
\]

Step 3: Evaluating the New Integral

The integral reduces to:

\[
I = \int_{0}^{1} y e^{y^2} \, dy
\]

Substitution Method

Let’s perform substitution:

- Set \(u = y^2 \)
- Then, \(du = 2y \, dy \)

Rearranged:

\[
y \, dy = \frac{du}{2}
\]

When \(y = 0 \):
\[
u = 0
\]

When \(y = 1 \):
\[
u = 1
\]

Rewriting the integral:

\[

$$
I = \int_{u=0}^{1} e^{u} \cdot \frac{1}{2} \, du = \frac{1}{2} \int_{0}^{1} e^{u} \, du
$$

Evaluating:

$$
I = \frac{1}{2} \left[e^{u} \right]_0^1 = \frac{1}{2} (e^{1} - e^{0}) = \frac{1}{2} (e - 1)
$$

Final Result and Confirmation

The value of the integral is:

$$
\boxed{
I = \frac{e - 1}{2}
}
$$

This matches the value obtained through the original order of integration, confirming the correctness of our process.

Broader Implications and Practical Tips

When to Change the Order of Integration

- Complex integrand: If the integrand simplifies in a different order.
- Complex limits: When current limits result in difficult integration.
- Geometric insight: Visualizing the region to find a more straightforward description.

Common Pitfalls

- Incorrect region description: Misidentifying the bounds leads to wrong limits.
- Neglecting variable dependence: Remember that the limits may depend on the other variable.
- Forgetting to adjust limits when switching order: Always re-derive the region bounds carefully.

Strategies for Effective Rewriting

- Sketch the region: Graph to understand the bounds.
- Express bounds in both variables: Practice converting between different descriptions.
- Use substitution wisely: Recognize integrand forms that suggest substitution.

Conclusion

Rewriting a double integral using a different order of integration is a powerful technique that can greatly simplify the evaluation process. Through careful analysis of the integration region, graphical visualization, and methodical limit transformation, complex integrals become manageable. The example provided demonstrates how changing the order led to a straightforward integral involving an exponential function, which was easily evaluated. Mastery of this skill enhances problem-solving flexibility and deepens comprehension of multivariable calculus, making it an essential tool for mathematicians, engineers, and applied scientists alike.

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About the Author

[Your Name] is a dedicated mathematics educator and researcher with extensive experience in calculus, mathematical analysis, and applied mathematics. With a focus on making complex concepts accessible, they have authored numerous articles and tutorials aimed at students and professionals seeking to deepen their understanding of calculus and its applications.

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