Show That The Given Curve C(t) Is A Flow Line Of The Given Velocity Vector Field F(x, Y, Z). C(t) = (t,

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Introduction to Flow Lines and Vector Fields

Understanding the behavior of particles moving within a velocity vector field is fundamental in fields such as fluid dynamics, physics, and engineering. A key concept in this context is the idea of flow lines, also known as streamlines, which represent the trajectories that particles follow as they move through the field. When analyzing whether a specific curve is a flow line of a given vector field, we utilize the mathematical relationship between the curve's parameterization and the velocity vector field.

This article provides an in-depth explanation of how to verify that a given curve C(t) is a flow line of a specified velocity vector field F(x, y, z). We will explore the theoretical background, step-by-step methodology, and illustrative examples to clarify this important concept.

Fundamentals of Flow Lines and Velocity Fields

What Is a Velocity Vector Field?

A velocity vector field F(x, y, z) assigns a velocity vector to every point in space. It is a vector function:

$$
F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle
$$

where P, Q, and R are component functions representing the velocity in the x, y, and z directions, respectively.

What Are Flow Lines?

Flow lines are curves in space that are tangent to the velocity vectors at every point. Formally, a curve C(t) = (x(t), y(t), z(t)) is a flow line of F if, at every point along the curve,

$$
\frac{d}{dt} C(t) = F\big(x(t), y(t), z(t) \big)
$$

This means the velocity vector of the particle moving along the curve at time t is exactly the same as the velocity vector field evaluated at the point C(t).

Mathematical Characterization of Flow Lines

Differential Equation Formulation

The condition for $C(t)$ being a flow line can be expressed as a system of first-order ordinary differential equations (ODEs):

$$
\frac{dx}{dt} = P(x, y, z)
$$

$$
\frac{dy}{dt} = Q(x, y, z)
$$

$$
\frac{dz}{dt} = R(x, y, z)
$$

Given a specific curve $C(t) = (x(t), y(t), z(t))$, to verify if it is a flow line, you need to check whether these equations are satisfied for all t in the domain of $C(t)$.

Flow Line Verification Criterion

To confirm that $C(t)$ is a flow line of F, perform the following steps:

1. Compute the derivatives of the curve components: $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

2. Evaluate the vector field at each point along the curve: $F(C(t)) = \langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \rangle$.

3. Compare the derivatives and the evaluated field: verify whether

$$
\frac{d}{dt} C(t) = F(C(t))
$$

for all t.

If this equality holds identically (or at least within the domain of interest), then $C(t)$ is a flow line of F.

Step-by-Step Procedure for Show That C(t) Is a Flow Line

1. Identify the Curve and Vector Field

Suppose we are given:

- A curve $C(t) = (x(t), y(t), z(t))$.

- A vector field $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$.

For example:

$$
C(t) = (t, y(t), z(t))
$$

and

$$
F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle
$$

with specific component functions.

2. Compute Derivatives of the Curve

Calculate:

$$
\frac{dx}{dt} = x'(t)
$$

$$
\frac{dy}{dt} = y'(t)
$$

$$
\frac{dz}{dt} = z'(t)
$$

for all t.

3. Evaluate the Vector Field Along the Curve

Substitute $(x(t), y(t), z(t))$ into the vector field:

$$
F(C(t)) = \left\langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \right\rangle
$$

This gives a vector function of t.

4. Verify the Equality of Derivatives and Field Evaluation

Check whether:

$$
x'(t) = P(x(t), y(t), z(t))
$$

$$
y'(t) = Q(x(t), y(t), z(t))
$$

$$
z'(t) = R(x(t), y(t), z(t))
$$

hold for all t. If they do, C(t) is a flow line.

Illustrative Example

Let's consider a concrete example to solidify the concept.

Given:

$$
C(t) = (t, t^2, 2t)
$$

and

$$
F(x, y, z) = \langle x, 2y, z \rangle
$$

Goal: Show that C(t) is a flow line of F.

Step 1: Compute derivatives of C(t)

$$
x(t) = t \Rightarrow x'(t) = 1
$$
$$
y(t) = t^2 \Rightarrow y'(t) = 2t
$$
$$
z(t) = 2t \Rightarrow z'(t) = 2
$$

Step 2: Evaluate the field along C(t)

$$
F(C(t)) = \langle x(t), 2 y(t), z(t) \rangle = \langle t, 2 t^2, 2t \rangle
$$

Step 3: Verify the equality

Compare derivatives with field evaluations:

$$
x'(t) = 1 \quad \text{and} \quad P(x, y, z) = x = t
$$
$$
y'(t) = 2t \quad \text{and} \quad Q(x, y, z) = 2 y = 2 t^2
$$
$$
z'(t) = 2 \quad \text{and} \quad R(x, y, z) = z = 2t
$$

Note that:

- $x'(t) = t \neq 1$, but evaluating at $x(t) = t$, the component $P(x, y, z) = x = t$. So, $x'(t) = P(x(t), y(t), z(t))$ if and only if $x'(t) = t$. But from the derivative, $x'(t) = 1$. These are equal only when $t=1$.

- Similarly, compare other components.

Correction: Based on the derivatives, the derivatives of the curve are:

$$
x'(t) = 1
$$

but the component of the vector field at the point is:

$$
P = x = t
$$

which only equals 1 when $t=1$. Thus, the curve is a flow line only at specific points unless the derivatives match identically.

Conclusion: To verify C(t) is a flow line for all t, the derivatives must satisfy the differential equations globally, which they do not in this case. Therefore, C(t) is only a flow line at points where the derivatives match, or we need to adjust the vector field or the curve.

Alternative: Suppose the vector field is:

$$
F(x, y, z) = \langle 1, 2t, 2 \rangle
$$

which matches the derivatives. Then, the curve is a flow line.

This example demonstrates the importance of verifying the differential equations component-wise.

Additional Considerations and Applications

Handling Non-Exact Matches

In practice, the derivatives may not match perfectly across the entire domain, implying the curve is not a flow line everywhere. Instead, it may be a flow line only at specific points or under certain conditions.

Applications of Flow Line Verification

- Fluid Mechanics: To analyze the path of particles in a fluid flow.

- Physics: Understanding trajectories under force fields.

- Engineering: Designing paths that follow specific

Frequently Asked Questions

What does it mean for a curve C(t) to be a flow line of a velocity vector field F(x, y, z)?

A curve C(t) is a flow line (or integral curve) of the vector field F if its tangent vector at each point equals the value of the field at that point, i.e., C'(t) = F(C(t)).

Given C(t) = (t, y(t), z(t)), how do we verify that it is a flow line of F(x, y, z)?

We compute the derivative C'(t) and check whether it equals F(C(t)) for all t, i.e., verify that C'(t) = F(t, y(t), z(t)).

What steps are involved in showing that C(t) = (t, y(t), z(t)) is a flow line for the vector field F(x, y, z)?

1. Find C'(t). 2. Evaluate F at C(t), i.e., F(t, y(t), z(t)). 3. Show that C'(t) = F(C(t)) holds for all t.

If C(t) = (t, y(t), z(t)) is a flow line of F, what differential equations must y(t) and z(t) satisfy?

They must satisfy y'(t) = F_2(C(t)) and z'(t) = F_3(C(t)), where F_2 and F_3 are the y- and z-components of F, respectively.

How does verifying that C'(t) = F(C(t)) help in understanding the behavior of the flow line?

It confirms that the curve follows the velocity field exactly, illustrating the trajectory of particles moving under the influence of F.

Can you provide an example of verifying a flow line for a specific vector field and curve?

Yes. For example, if F(x, y, z) = (y, -x, 0) and C(t) = (cos t, sin t, z_0), then C'(t) = (-sin t, cos t, 0). Since F(C(t)) = (sin t, -cos t, 0), checking whether C'(t) = F(C(t)) shows whether C(t) is a flow line—here, it is not, indicating the need to adjust the curve or field.

Why is it important to verify the flow line condition for different points along C(t)?

Because it ensures that the curve is an integral curve of the vector field throughout its domain, confirming it truly represents a flow line of F.

Additional Resources

Show That The Given Curve C(t) Is A Flow Line Of The Given Velocity Vector Field F(x, y, z)

Introduction

In the realm of vector calculus and fluid dynamics, understanding the relationship between a given vector field and its flow lines (or streamlines) is fundamental. Flow lines, also known as streamlines, are curves that are everywhere tangent to the velocity vector field at every point. They depict the trajectory that a particle would follow if placed in the flow at a specific instant, providing critical insights into the behavior of fluid motion, electromagnetic fields, and more.

This article investigates the process of demonstrating that a particular curve, specifically $C(t) = (t, y(t), z(t))$, is a flow line (or streamline) of a given velocity vector field $\mathbf{F}(x, y, z)$. Our focus will be on the rigorous mathematical framework that underpins this relationship, including the necessary conditions and proof strategies.

Theoretical Foundations

1. Vector Fields and Flow Lines

A vector field in three-dimensional space, $\mathbf{F}(x, y, z)$, assigns a vector to each point (x, y, z). It often models physical phenomena such as fluid velocity, magnetic fields, or force fields.

A flow line (or streamline) of $\mathbf{F}$ is a curve $\mathbf{C}(t) = (x(t), y(t), z(t))$ satisfying:

$$
\frac{d\mathbf{C}}{dt} = \mathbf{F}(\mathbf{C}(t))
$$

This differential equation stipulates that the tangent vector to the curve at any point matches the vector field at that point.

2. The Differential Equation of a Streamline

Given the parametric curve $C(t)$, the condition for it to be a streamline is:

$$
\frac{d}{dt} C(t) = \mathbf{F}(C(t))
$$

which expands component-wise to:

$$
\begin{cases}
x'(t) = F_1(x(t), y(t), z(t)) \\
y'(t) = F_2(x(t), y(t), z(t)) \\
z'(t) = F_3(x(t), y(t), z(t))
\end{cases}
$$

where $\mathbf{F} = (F_1, F_2, F_3)$.

Demonstrating that $C(t) = (t, y(t), z(t))$ Is a Flow Line

Suppose the curve is given explicitly as:

$$
C(t) = (t, y(t), z(t))
$$

\]

and the velocity field:

$$
\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))
$$

Our goal: to verify whether $C(t)$ is a flow line of $\mathbf{F}$.

Step-by-Step Process

1. Compute the Derivative $C'(t)$

Differentiating component-wise yields:

$$
C'(t) = \left(\frac{d}{dt} t, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right) = (1, y'(t), z'(t))
$$

---

2. Express $\mathbf{F}$ Along the Curve

Evaluate the vector field along $C(t)$:

$$
\mathbf{F}(C(t)) = \left(F_1(t, y(t), z(t)), F_2(t, y(t), z(t)), F_3(t, y(t), z(t)) \right)
$$

---

3. Set Up the Differential Equations

For $C(t)$ to be a flow line, the following must hold:

$$
\boxed{
(1, y'(t), z'(t)) = \left(F_1(t, y(t), z(t)), F_2(t, y(t), z(t)), F_3(t, y(t), z(t)) \right)
}
$$

This leads to the system:

$$
\begin{cases}
x' (t) = 1 = F_1(t, y(t), z(t)) \\
y' (t) = y'(t) = F_2(t, y(t), z(t)) \\
z' (t) = z'(t) = F_3(t, y(t), z(t))
\end{cases}
$$

Necessary Conditions for $C(t)$ to Be a Flow Line

The key step involves verifying two conditions:

1. Existence of $y(t)$, $z(t)$ satisfying the differential equations:

$$
y'(t) = f_2(t, y(t), z(t))
$$

$$
z'(t) = f_3(t, y(t), z(t))
$$

with initial conditions $y(t_0) = y_0$, $z(t_0) = z_0$.

2. Compatibility condition:

$$
f_1(t, y(t), z(t)) \equiv 1
$$

which implies that along the curve, the x-component of the vector field must be identically 1.

Application: Verifying the Conditions

Suppose the vector field is explicitly given, for example:

$$
\mathbf{F}(x, y, z) = (1, f_2(x, y, z), f_3(x, y, z))
$$

and the curve:

$$
C(t) = (t, y(t), z(t))
$$

To confirm $C(t)$ is a flow line:

- Step 1: Confirm $f_1(t, y(t), z(t)) = 1$ for all t.

- Step 2: Solve the system:

$$
\begin{cases}
y'(t) = f_2(t, y(t), z(t)) \\
z'(t) = f_3(t, y(t), z(t))
\end{cases}
$$

with initial conditions aligned with the initial point $C(t_0)$.

- Step 3: Check whether the solutions $y(t)$ and $z(t)$ satisfy these equations.

If all these conditions hold, then $C(t)$ is indeed a flow line of $\mathbf{F}$.

Illustrative Example

Given:

$$
\mathbf{F}(x, y, z) = (1, y, z)
$$

and

$$
C(t) = (t, y(t), z(t))
$$

Verification:

- Compute $C'(t) = (1, y'(t), z'(t))$.

- Evaluate $\mathbf{F}(C(t)) = (1, y(t), z(t))$.

- Comparing:

$$
(1, y'(t), z'(t)) \stackrel{?}{=} (1, y(t), z(t))
$$

which implies:

$$
\begin{cases}
y'(t) = y(t) \\
z'(t) = z(t)
\end{cases}
$$

Solving the systems:

$$
\begin{cases}
dy/dt = y(t) \Rightarrow y(t) = A e^{t} \\
dz/dt = z(t) \Rightarrow z(t) = B e^{t}
\end{cases}
$$

for constants A, B.

Conclusion:

- The curve $C(t) = (t, A e^{t}, B e^{t})$ is a flow line of $\mathbf{F}(x, y, z) = (1, y, z)$.

- If $y(t)$ and $z(t)$ are chosen as above, the differential equations are satisfied, confirming $C(t)$ is a flow line.

Deeper Insights and Common Pitfalls

1. Parameterization and Reparametrization

Not all flow lines are naturally parameterized by t in the form $(t, y(t), z(t))$. Often, flow lines can be reparameterized or written in different forms, but the key is the tangent condition:

\[
\frac{d\mathbf{C}}{dt} = \mathbf{F}(\mathbf{C}(t))
\]

The choice \(x(t) = t \) simplifies verification in many cases but is not a necessity.

2. Ensuring the Existence of Solutions

The existence and uniqueness theorems (e.g., Picard-Lindel� f) guarantee solutions to the differential system under certain regularity conditions of \(\mathbf{F} \). When demonstrating a curve is a flow line, it is crucial to verify initial conditions and the smoothness of \(\mathbf{F} \).

3. Limitations

- If \(F_1 \neq 1 \), then the curve \(C(t) = (t, y(t), z(t)) \) can only be a flow line if \(F_1

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