Solve The Following Using Laplace Transformation. Show All The Steps. No Other Method Will Be Accepted.

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When faced with differential equations in engineering, physics, or applied mathematics, the Laplace transformation often provides a powerful and systematic method to find solutions. This technique simplifies complex differential equations into algebraic equations in the Laplace domain, making them easier to solve. In this comprehensive guide, we will explore how to solve various differential equations step-by-step using Laplace transformations, adhering strictly to the specified method. Whether you are a student preparing for exams or a professional solving real-world problems, mastering this method is crucial for effective problem-solving.

Understanding Laplace Transformation: An Introduction

Before diving into solving differential equations, it's essential to understand what a Laplace transformation is and how it functions.

What is the Laplace Transform?

The Laplace transform is an integral transform that converts a function of time $f(t)$ (usually causal, meaning $f(t) = 0$ for $t < 0$) into a function of a complex variable s. The formal definition is:

$$
\boxed{
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st}f(t) \, dt
}
$$

Key points about the Laplace transform:

- It transforms differential equations into algebraic equations.
- It simplifies the process of handling initial conditions.
- It is particularly useful for linear ordinary differential equations with constant coefficients.

Basic Properties of Laplace Transforms

Understanding these properties is essential for solving differential equations efficiently:

- Linearity: $\mathcal{L}\{a f(t) + b g(t)\} = a F(s) + b G(s)$
- Derivatives: $\mathcal{L}\{f'(t)\} = s F(s) - f(0)$
- Integration: $\mathcal{L}\left\{\int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s}$
- Initial value theorem: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s)$

Step-by-Step Guide to Solving Differential Equations Using Laplace Transform

The process of solving differential equations via Laplace transforms involves several systematic steps:

Step 1: Take the Laplace Transform of Both Sides of the Differential Equation

- Apply the Laplace transform to each term.
- Use properties of the Laplace transform, especially the derivatives property, to convert derivatives into algebraic terms involving s.

Step 2: Incorporate Initial Conditions

- When transforming derivatives, include initial conditions $f(0), f'(0),$ etc.
- These initial conditions are crucial for correctly forming the algebraic equation.

Step 3: Solve the Resulting Algebraic Equation for $F(s)$

- Rearrange the algebraic equation to isolate $F(s)$.
- Simplify the expression as needed.

Step 4: Perform the Inverse Laplace Transform

- Use Laplace transform tables, partial fraction decomposition, or known inverse transforms to revert $F(s)$ back into $f(t)$.

Step 5: Write the Final Solution

- Present the solution $f(t)$ explicitly.
- Verify the solution by substituting back into the original differential equation if necessary.

Worked Examples: Solving Differential Equations Using Laplace Transform

To illustrate this method thoroughly, let’s go through multiple examples step by step.

Example 1: First-Order Differential Equation

Solve $y'(t) + 3 y(t) = 0$ with initial condition $y(0) = 2$.

Step 1: Take the Laplace transform

$$
\mathcal{L}\{y'(t)\} + 3 \mathcal{L}\{y(t)\} = 0
$$

Using the property $\mathcal{L}\{y'(t)\} = s Y(s) - y(0)$:

$$
s Y(s) - 2 + 3 Y(s) = 0
$$

Step 2: Solve for $Y(s)$

$$
(s + 3) Y(s) = 2
$$

$$
Y(s) = \frac{2}{s + 3}
$$

Step 3: Inverse Laplace Transform

Using the table, $\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-a t}$:

$$
y(t) = 2 e^{-3 t}
$$

Final solution:

$$
\boxed{
y(t) = 2 e^{-3 t}
}
$$

Example 2: Second-Order Differential Equation

Solve $y''(t) - 4 y'(t) + 5 y(t) = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$.

Step 1: Take the Laplace transform

$$
\mathcal{L}\{y''(t)\} - 4 \mathcal{L}\{y'(t)\} + 5 \mathcal{L}\{y(t)\} = 0
$$

Using properties:

$$
s^2 Y(s) - s y(0) - y'(0) - 4 [s Y(s) - y(0)] + 5 Y(s) = 0
$$

Plugging initial conditions:

$$
s^2 Y(s) - s (1) - 0 - 4 (s Y(s) - 1) + 5 Y(s) = 0
$$

Simplify:

$$
s^2 Y(s) - s - 4 s Y(s) + 4 + 5 Y(s) = 0
$$

Group $Y(s)$ terms:

$$
(s^2 - 4 s + 5) Y(s) = s - 4
$$

Solve for $Y(s)$:

$$
Y(s) = \frac{s - 4}{s^2 - 4 s + 5}
$$

Step 2: Simplify and find inverse Laplace

Complete the square in the denominator:

$$
s^2 - 4 s + 5 = (s - 2)^2 + 1
$$

Express numerator to match terms:

$$
Y(s) = \frac{s - 2 - 2}{(s - 2)^2 + 1} = \frac{s - 2}{(s - 2)^2 + 1} - \frac{2}{(s - 2)^2 + 1}
$$

Apply inverse Laplace transforms:

- $\mathcal{L}^{-1}\left\{\frac{s - a}{(s - a)^2 + b^2}\right\} = e^{a t} \cos b t$
- $\mathcal{L}^{-1}\left\{\frac{b}{(s - a)^2 + b^2}\right\} = e^{a t} \sin$

b t \)

Thus:

$$
y(t) = e^{2 t} \cos t - 2 e^{2 t} \sin t
$$

Final solution:

$$
\boxed{
y(t) = e^{2 t} (\cos t - 2 \sin t)
}
$$

Advantages of Using Laplace Transformation

Employing Laplace transforms to solve differential equations offers numerous benefits:

- Simplification: Converts differential equations into manageable algebraic equations.
- Initial Conditions: Incorporates initial conditions naturally into the solution process.
- Handling Discontinuities: Effectively manages functions with discontinuities or impulses.
- Systematic Approach: Provides a clear, step-by-step method suitable for linear differential equations.

Practical Tips for Solving Using Laplace Transform

To maximize efficiency and accuracy, consider these tips:

- Always verify the initial conditions before transforming.
- Use Laplace transform tables for common transforms to speed up inverse calculations.
- When encountering complex fractions, perform partial fraction decomposition.
- Be cautious with domain restrictions and convergence issues.
- Practice transforming derivatives and integrals to become familiar with properties.

Conclusion: Mastering Differential Equations via Laplace Transform

Solving differential equations using Laplace transformation is a foundational skill in applied mathematics, engineering, and physics. By following the structured steps—transforming the equation, solving in the s-domain, and then applying the inverse transform—you can efficiently and accurately find solutions to a wide range of problems. Remember, the key to mastery lies in practice: work through various problems, familiarize yourself with transform tables, and develop confidence in manipulating algebraic expressions in the Laplace domain.

Adhering strictly to the method ensures consistency and reliability, especially when other methods are unsuitable

Frequently Asked Questions

How do you apply Laplace transformation to solve the differential equation dy/dt + 3y = 6 with initial condition y(0) = 2?

First, take the Laplace transform of both sides: L{dy/dt} + 3L{y} = 6/L{1}. Using properties, L{dy/dt} = sY(s) - y(0). So, sY(s) - 2 + 3Y(s) = 6/s. Combine like terms: (s + 3)Y(s) = 6/s + 2. Solve for Y(s): Y(s) = (6/s + 2) / (s + 3). Simplify and then take the inverse Laplace transform to find y(t).

What are the steps to solve the second-order differential equation y'' + 4y = 0 using Laplace transforms?

1. Take Laplace transform of both sides: L{y''} + 4L{y} = 0. 2. Use the property: L{y''} = s^2Y(s) - s y(0) - y'(0). 3. Substitute initial conditions y(0) and y'(0). 4. Solve for Y(s): Y(s) = [s y(0) + y'(0)] / (s^2 + 4). 5. Take inverse Laplace transform to find y(t).

How do you handle initial conditions when solving a differential equation using Laplace transform?

Initial conditions are incorporated directly during the Laplace transform process. For derivatives, the transform involves initial values: L{dy/dt} = sY(s) - y(0), and for y'' it includes y(0) and y'(0). These are substituted into the algebraic equation before solving for Y(s).

Can you demonstrate solving a first-order differential equation with a step function input using Laplace transforms?

Yes. For example, solve dy/dt + y = u(t), where u(t) is the unit step. Take Laplace transforms: sY(s) - y(0) + Y(s) = 1/s. Assuming y(0) = 0, solve for Y(s): Y(s) = 1 / (s + 1). Inverse Laplace yields y(t) = e^{-t} for t ≥ 0.

What is the general procedure for solving integral equations using Laplace transforms?

Transform the integral equation into algebraic form by applying Laplace transform to each term. Use properties like $L\{\int_0^t f(\tau)d\tau\} = F(s)/s$. Solve the algebraic equation for the transformed unknown, then take the inverse Laplace transform to find the solution.

How do you solve a differential equation involving convolution using Laplace transforms?

Recall that Laplace transform turns convolution into multiplication: $L\{f g\} = F(s)G(s)$. Take Laplace transforms of both sides of the convolution equation, solve for the transforms, then apply inverse Laplace transform to find the solution in the time domain.

What are the common pitfalls to avoid when solving differential equations with Laplace transforms?

Common pitfalls include neglecting initial conditions, improperly handling transforms of derivatives, forgetting to simplify algebraic expressions, and not verifying the inverse Laplace transform. Always carefully apply the transform properties and check the solution by differentiation.

How do you verify the solution obtained from Laplace transform method?

Verify by substituting the inverse Laplace transformed solution into the original differential equation to ensure it satisfies the equation and initial conditions. Alternatively, differentiate the solution as needed and compare with the original differential equation.

Additional Resources

Solve The Following Using Laplace Transformation. Show All The Steps. No Other Method Will Be Accepted.

Introduction to Laplace Transformation

Laplace transformation is a powerful integral transform widely used in engineering, physics, and mathematics to convert differential equations into algebraic equations. Its primary advantage is simplifying the process of solving linear ordinary differential equations (ODEs), especially those with initial conditions. The core idea involves transforming a function of time, $f(t)$, into a complex frequency domain function, $F(s)$, making the differentiation and integration processes more manageable.

Key Concepts:

- The Laplace transform of a function $f(t)$ is defined as:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st}f(t) dt
$$
- The inverse Laplace transform converts $F(s)$ back into $f(t)$.
- Laplace transforms are particularly effective for solving linear differential equations with constant coefficients, especially when initial conditions are involved.

Why Use Only Laplace Transformation?

In many problem-solving contexts, alternative methods like the method of undetermined coefficients, variation of parameters, or direct integration are common. However, this specific task mandates using Laplace transformation exclusively. This restriction emphasizes:

- Mastery of the Laplace method and its systematic steps.
- Understanding the transformation process from differential equations to algebraic equations.
- Demonstrating the ability to handle initial conditions directly in the Laplace domain.
- Enhancing problem-solving rigor and clarity by avoiding alternative techniques.

General Procedure for Solving Differential Equations Using Laplace Transform

To tackle differential equations solely with Laplace transforms, follow these structured steps:

1. Write down the differential equation and initial conditions.
2. Apply the Laplace transform to both sides of the differential equation.
3. Use the properties of Laplace transforms for derivatives:
- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
- $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
- And so forth for higher derivatives.
4. Substitute initial conditions into the transformed equation.
5. Solve the resulting algebraic equation for $F(s)$.
6. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ to retrieve $f(t)$.
7. Verify the solution by differentiating $f(t)$ to ensure it satisfies the original differential equation and initial conditions.

Step-by-Step Example: Solving a Differential

Equation Using Laplace Transformation

Let's illustrate the process with a detailed example:

Problem Statement:
Solve the differential equation

$$
f''(t) + 3f'(t) + 2f(t) = 0
$$

with initial conditions:

$$
f(0) = 4, \quad f'(0) = 1
$$

Step 1: Write the differential equation and initial conditions

- Differential equation:

$$
f''(t) + 3f'(t) + 2f(t) = 0
$$

- Initial conditions:

$$
f(0) = 4, \quad f'(0) = 1
$$

Step 2: Apply Laplace transform to both sides

Recall:
- $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
- $\mathcal{L}\{f(t)\} = F(s)$

Transform the entire differential equation:

$$
\mathcal{L}\{f''(t)\} + 3 \mathcal{L}\{f'(t)\} + 2 \mathcal{L}\{f(t)\} = 0
$$

This becomes:

$$
[s^2F(s) - s \cdot 4 - 1] + 3[sF(s) - 4] + 2F(s) = 0
$$

Step 3: Substitute initial conditions and simplify

Plug in $f(0) = 4$ and $f'(0) = 1$:

$$
s^2F(s) - 4s - 1 + 3sF(s) - 12 + 2F(s) = 0
$$

Group terms with $F(s)$:

$$
(s^2F(s) + 3sF(s) + 2F(s)) - 4s - 1 - 12 = 0
$$

Factor $F(s)$:

$$
F(s)(s^2 + 3s + 2) - (4s + 13) = 0
$$

Step 4: Solve for $F(s)$

Rearranged:

$$
F(s)(s^2 + 3s + 2) = 4s + 13
$$

Note that:

$$
s^2 + 3s + 2 = (s + 1)(s + 2)
$$

Thus,

$$
F(s) = \frac{4s + 13}{(s + 1)(s + 2)}
$$

Step 5: Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$

Express $F(s)$ as partial fractions:

$$
\frac{4s + 13}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}
$$

Multiply both sides by $(s + 1)(s + 2)$:

$$
4s + 13 = A(s + 2) + B(s + 1)
$$

Set specific values for s:

- For $s = -1$:

$$
4(-1) + 13 = A(1) + B(0) \Rightarrow -4 + 13 = A \Rightarrow A = 9
$$

- For $s = -2$:

$$
4(-2) + 13 = A(0) + B(-1) \Rightarrow -8 + 13 = -B \Rightarrow 5 = -B \Rightarrow B = -5
$$

Therefore,

$$
F(s) = \frac{9}{s + 1} - \frac{5}{s + 2}
$$

Using known Laplace inverse transforms:

- $\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-a t}$

We find:

$$
f(t) = 9e^{-t} - 5e^{-2t}
$$

Final Solution and Verification

The solution to the original differential equation, respecting the initial conditions, is:

$$
\boxed{
f(t) = 9e^{-t} - 5e^{-2t}
}
$$

To verify, differentiate $f(t)$:

$$
f'(t) = -9e^{-t} + 10e^{-2t}
$$
$$
f''(t) = 9e^{-t} - 20e^{-2t}
$$

Substitute into the original differential equation:

$$
f''(t) + 3f'(t) + 2f(t) = (9e^{-t} - 20e^{-2t}) + 3(-9e^{-t} + 10e^{-2t}) + 2(9e^{-t} - 5e^{-2t})
$$

Simplify:

$$
9e^{-t} - 20e^{-2t} - 27e^{-t} + 30e^{-2t} + 18e^{-t} - 10e^{-2t} = 0
$$

Combine like terms:

$$
(9 - 27 + 18) e^{-t} + (-20 + 30 - 10) e^{-2t} = 0
$$
$$
(0) e^{-t} + (0) e^{-2t} = 0
$$

Confirmed that the solution satisfies the differential equation.

Additional Considerations and Tips

- Handling Initial Conditions: Always incorporate initial conditions early in the process to simplify algebraic expressions.
- Partial Fraction Decomposition: Essential for inverse transforms, especially when dealing with rational functions.
- Standard Laplace Transforms: Familiarity with common transforms such as $\mathcal{L}\{e^{at}\}$, $\mathcal{L}\{\sin bt\}$, $\mathcal{L}\{\cos bt\}$, etc., is crucial.
- Inverse Transforms: Use tables or known formulas, and practice partial fractions to simplify the process.
- Verification: Always differentiate

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