Solve The System By Using Elementary Row Operations On The Equations. Follow The Systematic Elimination

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Solving a system of linear equations is a fundamental skill in algebra and linear algebra, essential for understanding various scientific and engineering problems. One of the most systematic and effective methods for solving such systems is through the use of elementary row operations combined with the technique known as Gaussian elimination or systematic elimination. This approach transforms the system's augmented matrix into a form that makes the solutions readily apparent, typically the row echelon form or reduced row echelon form. By following a structured sequence of steps and applying elementary operations carefully, one can solve even complex systems efficiently and accurately.

Understanding the Basics of Elementary Row Operations

Before diving into the solution process, it is vital to understand what elementary row operations are and how they modify a system of equations represented in matrix form.

What Are Elementary Row Operations?

Elementary row operations are three types of operations that can be performed on the rows of a matrix without changing the solution set of the corresponding system of equations. These operations are:

- **Row swapping (interchanging two rows):** Swapping the positions of two rows in the matrix.
- **Row scaling (multiplying a row by a non-zero scalar):** Multiplying all elements in a row by a scalar different from zero.
- **Row addition (adding a multiple of one row to another):** Replacing a row by the sum of itself and a multiple of another row.

Importance of Elementary Row Operations

These operations are crucial because they preserve the solution set of the system while transforming the matrix into a simpler form. This simplification facilitates straightforward back-substitution to find solutions.

Representing the System as an Augmented Matrix

The first step in the systematic elimination process is to convert the system of linear equations into an augmented matrix.

Example of Conversion

Consider the system:

$$
\begin{cases}
2x + 3y - z = 5 \\
4x + 7y + 2z = 10 \\
-2x + y + 2z = -1
\end{cases}
$$

The augmented matrix form is:

$$
\left[\begin{array}{ccc|c}
2 & 3 & -1 & 5 \\
4 & 7 & 2 & 10 \\
-2 & 1 & 2 & -1
\end{array}\right]
$$

This matrix encapsulates all the information contained in the original equations, and elementary row operations can be applied directly to this matrix.

Step-by-Step Systematic Elimination Process

The goal is to transform the augmented matrix into a form where solutions are easily obtainable through back-substitution. The typical process involves converting the matrix into row echelon form, and then into reduced row echelon form.

Step 1: Establish a Leading 1 in the First Row, First Column

- Select the first row as the pivot row.
- If necessary, swap rows to bring a non-zero element into the pivot position.

- Normalize the pivot row by dividing by the leading coefficient to make it 1.

Example:

In the above matrix, the first element of row 1 is 2. Divide row 1 by 2:

$$
R_1 \leftarrow \frac{1}{2} R_1
$$

Result:

$$
\left[\begin{array}{ccc|c}
1 & 1.5 & -0.5 & 2.5 \\
4 & 7 & 2 & 10 \\
-2 & 1 & 2 & -1
\end{array}\right]
$$

Step 2: Eliminate Entries Below the Leading 1

- Use row operations to make all entries below the pivot zero.

Example:

- Subtract 4 times row 1 from row 2:

$$
R_2 \leftarrow R_2 - 4 R_1
$$

- Add 2 times row 1 to row 3:

$$
R_3 \leftarrow R_3 + 2 R_1
$$

Calculations:

- $R_2: (4, 7, 2, 10) - 4(1, 1.5, -0.5, 2.5) = (4-4, 7-6, 2+2, 10-10) = (0, 1, 0, 0)$

- $R_3: (-2, 1, 2, -1) + 2(1, 1.5, -0.5, 2.5) = (-2+2, 1+3, 2-1, -1+5) = (0, 4, 1, 4)$

Updated matrix:

$$
\left[\begin{array}{ccc|c}
1 & 1.5 & -0.5 & 2.5 \\
0 & 1 & 0 & 0 \\
0 & 4 & 1 & 4
\end{array}\right]
$$

\]

Step 3: Create a Leading 1 in the Second Row, Second Column

- The second row already has a 1 in the second position – this is your new pivot.
- Use row operations to eliminate entries below this pivot.

Example:

- Subtract 4 times row 2 from row 3:

$$
R_3 \leftarrow R_3 - 4 R_2
$$

Calculation:

- $R_3: (0, 4, 1, 4) - 4(0, 1, 0, 0) = (0, 4-4, 1-0, 4-0) = (0, 0, 1, 4)$

Updated matrix:

$$
\left[\begin{array}{ccc|c}
1 & 1.5 & -0.5 & 2.5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 4
\end{array}\right]
$$

Step 4: Back-Substitution and Further Elimination

- Now that the matrix is in row echelon form, proceed to reduce it to reduced row echelon form by eliminating above the pivots.

Example:

- Eliminate the 1.5 in row 1, second column:

$$
R_1 \leftarrow R_1 - 1.5 R_2
$$

Calculation:

- $R_1: (1, 1.5, -0.5, 2.5) - 1.5(0, 1, 0, 0) = (1, 1.5-1.5, -0.5-0, 2.5-0) = (1, 0, -0.5, 2.5)$

- Eliminate the -0.5 in row 1, third column:

\[
R_1 \leftarrow R_1 + 0.5 R_3
\]

Calculation:

- \(R_1: (1, 0, -0.5, 2.5) + 0.5(0, 0, 1, 4) = (1, 0, -0.5+0.5, 2.5+2) = (1, 0, 0, 4.5) \)

- Finally, eliminate the 4 in row 3, last step:

- Use row 3 as the pivot row for the third variable, which is already in simplified form.

Interpreting the Final Matrix and Extracting Solutions

Once the matrix is in reduced row echelon form, solutions can be directly read off as:

\[
x = \text{value from row 1} \\
y = \text{value from row 2} \\
z = \text{value from row 3}
\]

Example:

From the last matrix:

\[
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 4.5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 4
\end{array}\right]
\]

The solutions are:

\[
x = 4.5, \quad y = 0, \quad z = 4
\]

Key Points to Remember in Systematic Elimination

Strategies for Effective Elimination

1. Always aim to create leading 1s in each pivot position.

2. Use row operations to systematically eliminate variables below and above the pivots.

3. Be cautious to maintain the correctness of each step, especially when multiplying or adding rows.

4. When encountering a zero pivot, consider swapping rows to find a non-zero pivot element.