Solve Trigonometric Equations Requiring The Quadratic Formula (Give All Answers, In (0,2), In Radians,

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Solving trigonometric equations is a fundamental skill in mathematics, particularly in calculus, physics, and engineering. Some equations involve quadratic expressions of trigonometric functions, requiring the use of the quadratic formula to find solutions. When tackling these equations, it is essential to understand how to manipulate the equations into quadratic form, apply the quadratic formula correctly, and interpret the solutions within the specified interval—in this case, between 0 and 2 radians. This comprehensive guide will walk you through the process of solving such equations, providing step-by-step solutions, key concepts, and tips to master this topic.

Understanding the Quadratic Form in Trigonometric Equations

Many trigonometric equations can be rewritten into a quadratic form by substitution. For example, equations involving $\sin \theta$ or $\cos \theta$ can often be expressed as quadratic equations in terms of these functions.

Common Patterns Leading to Quadratic Equations

- Equations like $\sin^2\theta + a \sin \theta + b = 0$
- Equations like $\cos^2\theta + c \cos \theta + d = 0$
- Mixed forms that can be converted into quadratic form through substitution

Why Use the Quadratic Formula?

The quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provides a direct method to solve quadratic equations when factoring is difficult or impossible. When the quadratic form appears in trigonometric equations, applying this formula helps find all possible solutions for the trigonometric functions, which we then convert back to the original variable, θ.

Step-by-Step Process for Solving Trigonometric Equations Using the Quadratic Formula

1. Rewrite the equation in terms of a single trigonometric function
2. Express the equation in quadratic form
3. Identify coefficients a, b, and c
4. Calculate the discriminant $D = b^2 - 4ac$
5. Determine the nature of solutions based on the discriminant
6. Apply the quadratic formula to find the solutions for the trigonometric function
7. Solve for θ within the interval (0, 2) radians
8. Verify solutions and discard extraneous roots outside the interval

Let's explore each step in detail with examples.

Example 1: Solving a Basic Trigonometric Quadratic Equation

Suppose we want to solve:

$$\sin^2 \theta - \sin \theta - 2 = 0 \quad \text{for} \quad \theta \in (0, 2)$$

Step 1: Substitution
Let $x = \sin \theta$. The equation becomes:

$$x^2 - x - 2 = 0$$

Step 2: Quadratic form
Already in quadratic form with $a=1$, $b=-1$, $c=-2$.

Step 3: Calculate discriminant

$$D = (-1)^2 - 4 \times 1 \times (-2) = 1 + 8 = 9$$

Since $D>0$, there are two real solutions.

Step 4: Apply quadratic formula

$$x = \frac{-(-1) \pm \sqrt{9}}{2 \times 1} = \frac{1 \pm 3}{2}$$

Solutions:

- $x = \frac{1 + 3}{2} = 2$
- $x = \frac{1 - 3}{2} = -1$

Step 5: Interpret solutions for $\sin \theta$

Recall that \(\sin θ \) must be in [-1, 1].

- \(\sin θ = 2 \) → Not possible, since sine cannot be greater than 1.
- \(\sin θ = -1 \) → Possible.

Step 6: Find θ in (0, 2) for \(\sin θ = -1 \)

\[
\sin θ = -1 \quad \Rightarrow \quad θ = \frac{3π}{2} \approx 4.712 \text{ radians}
\]

But we are asked for solutions in (0, 2) radians. Since \(\frac{3π}{2} \approx 4.712 \), which exceeds 2 radians, no solutions within the interval.

Result: No solutions in (0, 2) radians for the given equation.

Example 2: Solving a Cosine-Based Equation

Let's consider:

\[2 \cos^2 θ - 3 \cos θ + 1 = 0 \quad \text{for} \quad θ \in (0, 2) \]

Step 1: Substitution
Let \(x = \cos θ \). The quadratic becomes:

\[2x^2 - 3x + 1 = 0 \]

Step 2: Identify coefficients
\(a=2 \), \(b=-3 \), \(c=1 \).

Step 3: Calculate discriminant
\[D = (-3)^2 - 4 \times 2 \times 1 = 9 - 8 = 1 \]

Step 4: Find roots
\[
x = \frac{3 \pm \sqrt{1}}{2 \times 2} = \frac{3 \pm 1}{4}
\]

Solutions:

- \(x = \frac{3 + 1}{4} = 1 \)
- \(x = \frac{3 - 1}{4} = \frac{1}{2} \)

Step 5: Solutions for \(\cos θ \)

- \(\cos θ = 1 \)

- $\cos θ = \frac{1}{2}$

Step 6: Find θ in (0, 2)

- For $\cos θ = 1$:

$$
\cos θ = 1 \quad \Rightarrow \quad θ = 0
$$

But 0 is not in the interval (0, 2), so discard.

- For $\cos θ = \frac{1}{2}$:

$$
θ = \frac{π}{3} \approx 1.047 \text{ radians}
$$
and

$$
θ = 2π - \frac{π}{3} = \frac{5π}{3} \approx 5.236 \text{ radians}
$$

Since the interval is (0, 2), only $θ = \frac{π}{3}$ ≈ 1.047 radians fits.

Result: The only solution in (0, 2) radians is approximately 1.047 radians.

Special Considerations When Using the Quadratic Formula

- Discriminant Analysis:
- $D > 0$ indicates two real solutions.
- $D = 0$ indicates one real solution.
- $D < 0$ indicates no real solutions for the trigonometric function, hence no solutions in the interval.

- Range of Trigonometric Functions:
Always verify whether solutions for $\sin θ$ or $\cos θ$ lie within [-1, 1]. Solutions outside this range are extraneous and must be discarded.

- Interval of Solutions:
Since the problem restricts solutions to (0, 2) radians, convert all solutions, including those involving inverse trigonometric functions, within this interval. Remember, the general solutions for sine and cosine are periodic, so consider all possible solutions within the interval.

General Strategy for Complex Equations

1. Convert the equation into quadratic form using substitution.
2. Calculate the discriminant and analyze the roots.
3. For each valid root within [-1, 1], find the corresponding θ using inverse trig functions.
4. Use the periodicity of sine and cosine to find all solutions within (0, 2).
5. Discard extraneous solutions outside the interval.

Additional Examples and Practice Problems

To solidify understanding, here are practice problems:

- Solve $3 \sin^2 \theta - 2 \sin \theta - 1 = 0$ for $\theta \in (0, 2)$.
- Solve $\cos^2 \theta + \cos \theta - 2 = 0$ for $\theta \in (0, 2)$.
- Determine all solutions for $4 \sin^2 \theta - 3 = 0$ in $(0, 2)$.

Working through these problems using the methods detailed above will enhance your ability to solve quadratic trigonometric equations efficiently and accurately.

Conclusion

Solving trigonometric equations involving quadratic expressions requires a systematic approach—rewriting the equations in quadratic form, applying the quadratic formula, and carefully analyzing the solutions within the specific interval. Remember to verify solutions against the range of the trigonometric functions and to consider the periodic nature of

Frequently Asked Questions

Solve the equation $\cos^2(x) - 3\cos(x) + 2 = 0$ for x in $(0, 2\pi)$.

$x = \pi/3, 5\pi/3$

Find all solutions to $2\sin^2(x) - \sin(x) - 1 = 0$ in the interval $(0, 2\pi)$.

$x = \pi/6, 5\pi/6$

Solve the quadratic in cos(x): $4\cos^2(x) - 4\cos(x) + 1 = 0$ for x in $(0, 2\pi)$.

$x = \pi/4, 3\pi/4$

Determine all solutions for $3\tan^2(x) - 2\tan(x) - 1 = 0$ in $(0, 2\pi)$.

$x = \pi/6, 5\pi/6$

Solve for x in $(0, 2\pi)$: $2\sec^2(x) - 3 = 0$.

$x = \pi/3, 2\pi/3$

Find solutions to $5\sin^2(x) - 3\sin(x) = 0$ in $(0, 2\pi)$.

$x = 0, \pi/6, 5\pi/6, \pi$

Solve the quadratic in tan(x): $\tan^2(x) - 4\tan(x) + 3 = 0$ for x in $(0, 2\pi)$.

$x = \pi/4, 3\pi/4$

Find all solutions to $\cos^2(x) + \cos(x) - 2 = 0$ in $(0, 2\pi)$.

$x = 2\pi/3, 4\pi/3$

Solve $2\sin^2(x) + \sin(x) - 1 = 0$ for x in $(0, 2\pi)$.

$x = \pi/6, 5\pi/6$

Determine solutions to $3\sec^2(x) - 4 = 0$ in $(0, 2\pi)$.

$x = \pi/3, 2\pi/3$

Additional Resources

Solve Trigonometric Equations Requiring The Quadratic Formula (Give All Answers, In (0,2), In Radians)

Introduction: The Intersection of Trigonometry and Algebra

Solve trigonometric equations requiring the quadratic formula represents a fascinating confluence of algebraic techniques and trigonometric identities. This approach is fundamental in advanced mathematics, physics, engineering, and computer science, where problems often manifest in forms that demand both algebraic manipulation and a deep understanding of trigonometric functions. The challenge lies in transforming complex trigonometric equations into algebraic quadratic equations, which can then be solved using the quadratic formula. This method not only simplifies problem-solving but also broadens the understanding of the behavior of trigonometric functions within specific domains, especially when solutions are constrained within the interval $(0, 2\pi)$ radians.

In this article, we explore the methodology for solving such equations comprehensively. We will analyze common types of equations, demonstrate step-by-step solutions, and discuss the significance of each step, ensuring a clear understanding of the entire process. This approach is crucial for students aiming to master trigonometry and algebra, as well as professionals engaged in complex analytical tasks.

Understanding the Quadratic Nature of Trigonometric Equations

Why Do Some Trigonometric Equations Reduce to Quadratics?

Certain trigonometric equations inherently take on a quadratic form when expressed in terms of a single trigonometric function. For instance, equations involving sine or cosine squared naturally lead to quadratic equations because of the Pythagorean identities:

- $\sin^2 \theta + \cos^2 \theta = 1$
- $1 - \sin^2 \theta = \cos^2 \theta$
- $1 - \cos^2 \theta = \sin^2 \theta$

When the original problem involves squared terms or products of sine and cosine, algebraic substitution often transforms the equation into a quadratic form, making the quadratic formula applicable.

Example:

Solve for θ in the equation:

$$\sin^2 \theta + 3 \sin \theta - 4 = 0$$

This is a quadratic in $\sin \theta$, which simplifies the process significantly.

General Form of Trigonometric Quadratic Equations

The typical structure of these equations can be written as:

$$a \cdot \sin^2 \theta + b \cdot \sin \theta + c = 0$$
or
$$a \cdot \cos^2 \theta + b \cdot \cos \theta + c = 0$$

where a, b, and c are real coefficients. The goal is to:

1. Recognize the quadratic form.
2. Substitution: Let $x = \sin \theta$ or $x = \cos \theta$.
3. Solve for x using the quadratic formula:

$$x = \frac{ -b \pm \sqrt{b^2 - 4ac} }{2a}$$

4. Back-substitute to find θ.

Important: When solutions for x are outside the range $[-1, 1]$, they are extraneous because $\sin \theta$ and $\cos \theta$ only take values within this interval.

Step-by-Step Solution Process

1. Transform the Equation into a Quadratic Form

Identify the form and perform any necessary substitutions. For example, if the equation involves $\sin^2 \theta$, replace it with a variable x:

$$x = \sin \theta$$

Rearranged into a standard quadratic:

$$ a x^2 + b x + c = 0 $$

2. Apply the Quadratic Formula

Calculate the discriminant:

$$ D = b^2 - 4ac $$

- If $D < 0$, the quadratic has no real solutions, and thus no solutions for θ.
- If $D \geq 0$, find the roots:

$$ x_{1,2} = \frac{ -b \pm \sqrt{D} }{2a} $$

3. Verify Solutions within the Domain

Since $\sin \theta$ and $\cos \theta$ are bounded between -1 and 1, discard any solutions where x is outside this interval.

4. Find the Corresponding Angles

Solve for θ:

$$ \theta = \arcsin x \quad \text{or} \quad \theta = \arccos x $$

Remember the principal values and general solutions:

- For $\sin \theta = x$:

$$ \theta = \arcsin x + 2k\pi \quad \text{or} \quad \theta = \pi - \arcsin x + 2k\pi $$

- For $\cos \theta = x$:

$$ \theta = \arccos x + 2k\pi \quad \text{or} \quad \theta = -\arccos x + 2k\pi $$

Within the domain $(0, 2\pi)$, identify the specific solutions by evaluating these expressions.

Practical Examples and Detailed Solutions

Example 1: Solving $\sin^2 \theta + 3 \sin \theta - 4 = 0$ in (0, 2π)

Step 1: Recognize the quadratic form

Let $x = \sin \theta$, then:

$$ x^2 + 3x - 4 = 0 $$

Step 2: Apply quadratic formula

$$ D = 3^2 - 4 \times 1 \times (-4) = 9 + 16 = 25 $$

$$ x = \frac{ -3 \pm \sqrt{25} }{2} = \frac{ -3 \pm 5 }{2} $$

Calculate roots:

- $x_1 = \frac{ -3 + 5 }{2} = \frac{2}{2} = 1$
- $x_2 = \frac{ -3 - 5 }{2} = \frac{ -8 }{2} = -4$

Step 3: Verify within the domain

- $x_1 = 1$ is valid for $\sin \theta$.
- $x_2 = -4$ is outside $[-1, 1]$, discard.

Step 4: Find θ

$\sin \theta = 1$

- $\theta = \arcsin 1 = \pi/2$

In $(0, 2\pi)$, this corresponds to:

- $\boxed{\theta = \pi/2}$

No other solutions, because $\sin \theta = 1$ only at $\pi/2$ in the principal range.

Example 2: Solving $2 \cos^2 \theta - \cos \theta - 1 = 0$ in (0, 2π)

Step 1: Recognize the quadratic

Let $x = \cos \theta$:

$$ 2x^2 - x - 1 = 0 $$

Step 2: Quadratic formula

Calculate discriminant:

$$ D = (-1)^2 - 4 \times 2 \times (-1) = 1 + 8 = 9 $$

Roots:

$$ x = \frac{1 \pm \sqrt{9}}{2 \times 2} = \frac{1 \pm 3}{4} $$

- $ x_1 = \frac{1 + 3}{4} = \ \frac{4}{4} = 1 $
- $ x_2 = \frac{1 - 3}{4} = \frac{-2}{4} = -\frac{1}{2} $

Step 3: Verify

- $x_1 = 1$, valid for $\cos \theta$.
- $x_2 = -1/2$, valid for $\cos \theta$.

Step 4: Find θ

- For $\cos \theta = 1$:

$\theta = 0$

- For $\cos \theta = -1/2$:

$\theta = \arccos (-1/2) = 2\pi/3$ or $4\pi/3$

Within $(0, 2\pi)$, solutions are:

$$
\boxed{
\theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}
}
$$

Special Considerations and Constraints

While solving these equations, certain factors must be carefully considered to ensure solutions are valid within the specified domain:

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- Extraneous solutions: Roots outside \([-1, 1]\) are invalid because sine and cosine cannot produce such values
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