The Circle In Which The Sphere Of Radius 3 Centered At The Origin Intersects With The Plane Through The

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Understanding the intersection of a sphere and a plane is a fundamental concept in geometry that has applications across various fields, including computer graphics, physics, engineering, and mathematics. In this detailed exploration, we focus on a specific scenario: a sphere of radius 3 centered at the origin and its intersection with an arbitrary plane passing through the space. Our goal is to analyze the nature of this intersection, determine the conditions under which it occurs, and understand the resulting geometric shape.

Defining the Sphere and the Plane

The Sphere Centered at the Origin

A sphere in three-dimensional space is defined as the set of all points equidistant from a fixed point, called the center. For our case:

- Center: $O = (0, 0, 0)$
- Radius: $r = 3$

The equation of this sphere is:

$$
x^2 + y^2 + z^2 = 9
$$

This equation represents all points (x, y, z) in $\mathbb{R}^3$ that are exactly 3 units away from the origin.

The Plane Passing Through the Space

A plane in three-dimensional space can be described generally by a linear equation:

$$
A x + B y + C z + D = 0
$$

where A, B, C are the coefficients that define the orientation of the plane, and D determines its position relative to the origin.

Since the plane passes through a specific point, say $P = (x_0, y_0, z_0)$, the equation becomes:

$$
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0
$$

or equivalently,

$$
A x + B y + C z = - (A x_0 + B y_0 + C z_0)
$$

In our scenario, the key condition is that the plane passes through the origin, which simplifies the equation to:

$$
A x + B y + C z = 0
$$

since substituting $(0, 0, 0)$ yields $0 = 0$.

Geometric Nature of the Intersection

Possible Intersections Between Sphere and Plane

The intersection of a sphere and a plane can be:

- **No intersection** (the plane is outside the sphere)
- **A single point** (the plane is tangent to the sphere)
- **A circle** (the plane cuts through the sphere)

Our focus is on the last case, where the plane intersects the sphere in a circle, often called the "circle of intersection" or "circle in the sphere."

Conditions for the Intersection to be a Circle

Given the sphere $x^2 + y^2 + z^2 = 9$ and a plane $A x + B y + C z + D = 0$, the intersection will be a circle provided that the plane actually intersects the sphere, i.e., the distance from the sphere's center to the plane is less than the radius.

The distance d from the center $O=(0,0,0)$ to the plane is:

$$
d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}
$$

- If $d > r$, no intersection.
- If $d = r$, tangent point (single point).
- If $d < r$, intersection is a circle.

Since the plane passes through the origin, $D=0$, and the distance simplifies to:

$$
d= 0
$$

which means the plane passes through the center of the sphere, and the intersection is a circle of maximum size.

Deriving the Equation of the Intersection Circle

Substituting the Plane Equation into the Sphere Equation

When the plane passes through the origin, its equation is:

$$
A x + B y + C z = 0
$$

To find the intersection, we can parameterize or eliminate variables to express the intersection explicitly.

Method 1: Using Orthogonal Projection

The intersection of the sphere and the plane is a circle lying within the plane. The steps are:

1. Recognize that the intersection circle lies in the plane $A x + B y + C z = 0$.

2. Find the intersection radius r_{circle}:

$$
r_{\text{circle}} = \sqrt{r^2 - d^2}
$$

But since $d=0$ (the plane passes through the origin), the circle's radius is:

$$
r_{\text{circle}} = r = 3
$$

3. The circle is centered at the projection of the sphere’s center onto the plane, which is the origin itself because the center is on the plane. Thus, the circle’s center in the plane is at $(0,0,0)$.

4. To write the circle explicitly, choose an orthonormal basis for the plane.

Method 2: Explicit Equation of the Circle

To explicitly write the circle, proceed as follows:

1. The plane passes through the origin, so its equation is $A x + B y + C z = 0$.

2. The intersection is the set of points satisfying:

$$
x^2 + y^2 + z^2 = 9
$$
and
$$
A x + B y + C z = 0
$$

3. Find a parametric form:

- Find a set of two orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ lying in the plane, forming an orthonormal basis for the plane.

- The general point on the circle can be written as:

$$
\mathbf{P} = \mathbf{0} + r_{\text{circle}} (\cos \theta \mathbf{u} + \sin \theta \mathbf{v})
$$

where $\theta \in [0, 2\pi)$.

Explicit Representation of the Intersection Circle

Choosing an Orthonormal Basis for the Plane

Given the plane $A x + B y + C z=0$, the normal vector is:

$$
\mathbf{n} = (A, B, C)
$$

To find two vectors orthogonal to $\mathbf{n}$:

1. Pick an arbitrary vector not parallel to $\mathbf{n}$, for example:

- If $A \neq 0$, choose:

$$
\mathbf{u} = (-B, A, 0)
$$

and normalize it:

$$
\mathbf{u} = \frac{1}{\sqrt{A^2 + B^2}} (-B, A, 0)
$$

2. Find $\mathbf{v}$ as the cross product:

\[

\mathbf{v} = \mathbf{n} \times \mathbf{u}
\]

which guarantees \(\mathbf{v}\) is orthogonal to both \(\mathbf{n}\) and \(\mathbf{u}\).

3. Normalize \(\mathbf{v}\).

Once \(\mathbf{u}\) and \(\mathbf{v}\) are obtained, the circle of intersection is:

\[
\mathbf{P}(\theta) = r \left(\cos \theta \mathbf{u} + \sin \theta \mathbf{v} \right)
\]

with \(r=3\).

Final Equation of the Circle

Expressed parametrically:

\[
\boxed{
\begin{aligned}
x(\theta) &= r \left(u_x \cos \theta + v_x \sin \theta \right) \\
y(\theta) &= r \left(u_y \cos \theta + v_y \sin \theta \right) \\
z(\theta) &= r \left(u_z \cos \theta + v_z \sin \theta \right)
\end{aligned}
}
\]

for \(\theta \in [0, 2\pi)\).

Special Cases and Geometric Interpretations

Plane Passing Through the Center of the Sphere

As established, when the plane passes through the sphere's center, the intersection is a circle with radius equal to the sphere's radius (3 units). The circle lies entirely within the plane, centered at the origin.

Plane Not Passing Through the Center

If the plane does not pass through the origin, then:

- The distance from the sphere center to the plane is non-zero.
- The intersection, if it exists, is a circle with a radius less than 3.
- The circle's center is the projection of the sphere's center onto the plane, offset by the distance d.

The general radius:

$$
r_{\text{circle}} = \sqrt{r^2 - d^2}
$$

and the circle is centered

Frequently Asked Questions

What is the equation of the sphere centered at the origin with radius 3?

The equation of the sphere is $x^2 + y^2 + z^2 = 9$.

How do you find the intersection of the sphere with a plane passing through a point?

Substitute the plane's equation into the sphere's equation to find the intersection curve, which is typically a circle or empty/point if they are tangent.

What is the general form of a plane passing through a specific point?

A plane passing through point (x_0, y_0, z_0) with normal vector (A, B, C) is given by $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

How can I determine if the intersection of the sphere and the plane is a circle?

If the plane intersects the sphere in more than one point and is not tangent, the intersection is a circle. You can verify by solving the equations and checking the resulting shape.

What is the maximum distance a plane can be from the origin to still intersect the sphere of radius 3?

The plane's distance from the origin must be less than or equal to the radius of the sphere (3) for an intersection to occur. If the distance equals 3, the plane is tangent; less than 3, it intersects in a circle.

How do you find the radius of the circle formed by the intersection?

If the plane's distance from the origin is d, then the radius r of the intersection circle is $r = \sqrt{9 - d^2}$.

Can the intersection of the sphere and a plane through the origin be a single point?

Yes, if the plane is tangent to the sphere, the intersection is a single point, specifically at the point of tangency.

Additional Resources

The Circle In Which The Sphere Of Radius 3 Centered At The Origin Intersects With The Plane Through The

Understanding the intersection of a sphere and a plane is a fundamental concept in solid geometry with applications spanning from computer graphics and physics to engineering and mathematics. This detailed review explores the geometric relationships, mathematical formulations, and implications of the intersection between a sphere of radius 3 centered at the origin and an arbitrary plane passing through the same space.

Introduction to the Geometric Setup

Before delving into the specifics, it's crucial to establish the basic elements involved:

- Sphere: A perfectly symmetrical three-dimensional object defined by a center point and a radius.
- Plane: A flat, two-dimensional surface extending infinitely in all directions in three-dimensional space.
- Intersection: The set of points common to both the sphere and the plane.

In this particular case, we analyze:

- A sphere with center at the origin $O(0, 0, 0)$.
- A radius of 3 units.
- An arbitrary plane passing through the space, with the key condition that it passes through the origin.

Mathematical Representation of the Sphere

The sphere S with radius $r = 3$ centered at the origin is mathematically expressed as:

$$
x^2 + y^2 + z^2 = 9
$$

This equation describes all points (x, y, z) in three-dimensional space that are exactly 3 units away from the origin.

Equation of a Plane Passing Through the Origin

Any plane passing through the origin can be represented as:

$$
A x + B y + C z = 0
$$

where A, B, C are real numbers not all zero. These coefficients form the normal vector $\vec{n} = (A, B, C)$, which is perpendicular to the plane.

Key points:

- The plane passes through the origin, so the equation does not include a constant term.
- The normal vector $\vec{n}$ defines the orientation of the plane.

Intersection of Sphere and Plane: Geometric

Insight

The intersection of the sphere and the plane depends on the relative positioning of the plane:

- If the plane does not intersect the sphere: No points are common.
- If the plane is tangent to the sphere: The intersection reduces to a single point.
- If the plane cuts through the sphere: The intersection is a circle.

Given the sphere's symmetry and the plane passing through the origin, the intersection is always a circle when they intersect in more than one point.

Deriving the Intersection: Mathematical Approach

To find the intersection explicitly, substitute the plane's equation into the sphere's equation.

Step 1: Express z in terms of x and y:

$$
A x + B y + C z = 0 \Rightarrow z = -\frac{A x + B y}{C}, \quad
\text{assuming } C \neq 0
$$

If $C = 0$, then the plane is vertical with respect to the z-axis, and a similar approach applies.

Step 2: Substitute z into the sphere's equation:

$$
x^2 + y^2 + \left(-\frac{A x + B y}{C} \right)^2 = 9
$$

Simplify:

$$
x^2 + y^2 + \frac{(A x + B y)^2}{C^2} = 9
$$

Step 3: Expand and rearrange:

$$
x^2 + y^2 + \frac{A^2 x^2 + 2 A B x y + B^2 y^2}{C^2} = 9
$$

\]

Multiply through by C^2:

$$
C^2 x^2 + C^2 y^2 + A^2 x^2 + 2 A B x y + B^2 y^2 = 9 C^2
$$

Group similar terms:

$$
(A^2 + C^2) x^2 + (B^2 + C^2) y^2 + 2 A B x y = 9 C^2
$$

This is the equation of a conic in x and y, which, under normal circumstances, is a circle.

Step 4: Recognize the intersection as a circle:

- The general quadratic form above describes a circle if the discriminant condition is satisfied.
- Complete the square or rotate axes to find the circle's center and radius.

Determining the Nature of the Intersection

The key quantity governing the intersection's existence and size is the distance from the sphere's center to the plane.

Distance from the origin to the plane:

$$
d = \frac{|A \times 0 + B \times 0 + C \times 0|}{\sqrt{A^2 + B^2 + C^2}} = 0
$$

Since the plane passes through the origin, this distance is zero, which simplifies the analysis.

Implication:

- Because the plane passes through the origin, the intersection points lie on both the sphere and the plane that contains the origin.

Radius of the intersection circle:

- The intersection circle's radius r_{circle} is related to the sphere's radius $R = 3$ by the relation:

$$
r_{circle} = \sqrt{R^2 - d^2}
$$

- With $d = 0$, the circle's radius simplifies to:

$$
r_{circle} = \sqrt{9 - 0} = 3
$$

Hence, the intersection is a great circle of the sphere with radius 3.

Special Cases and Variations

While the above derivation assumes the plane passes through the origin, variations occur if the plane does not pass through the origin.

1. Plane Not Passing Through the Origin

The general plane:

$$
A x + B y + C z + D = 0
$$

- The distance from the sphere's center (origin) to the plane is:

$$
d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}
$$

- The intersection is:

- Empty if $d > R$ (the plane is outside the sphere).
- A single point if $d = R$ (tangent plane).
- A circle with radius:

$$
r_{circle} = \sqrt{R^2 - d^2}
$$

2. Orientation and Positioning of the Plane

- The angle between the plane and the sphere's center impacts the shape and size of the intersection.
- The normal vector $\vec{n}$ determines the plane's orientation.

Geometric Visualization and Implications

Visualizing the intersection helps in understanding the spatial relationships:

- When the plane passes through the sphere's center, the intersection is a great circle with maximum radius equal to the sphere's radius (here, 3 units).
- If the plane is offset but still intersects the sphere, the intersection is a smaller circle.
- If the plane is outside the sphere ($d > R$), there is no intersection.

Applications and Significance

Understanding the intersection of spheres and planes is crucial across multiple fields:

- Computer Graphics: Rendering objects and understanding clipping planes.
- Physics: Analyzing fields and wavefronts intersecting with boundaries.
- Engineering: Designing spherical components and their interactions with planar surfaces.
- Mathematics: Solving problems involving conic sections, spatial reasoning, and analytical geometry.

Summary of Key Formulas and Concepts

- Sphere Equation:

$$
x^2 + y^2 + z^2 = R^2
$$

- Plane Equation (through the origin):

$$
A x + B y + C z = 0
$$

- Distance from center to plane:

$$
d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}
$$

- Radius of intersection circle:

$$
r_{circle} = \sqrt{R^2 - d^2}
$$

- When $d = 0$, the intersection is a great circle with radius equal to the sphere's radius.

Conclusion

The intersection of a sphere of radius 3 centered at the origin with a plane passing through the same point is a rich geometric scenario that exemplifies the elegance of three-dimensional geometry. When the plane passes through the origin, the intersection is always a circle, specifically a great circle with radius 3. Variations in the plane's position and orientation significantly influence the shape and size of the intersection, governed primarily by the distance from the sphere's center to the plane.

This geometric relationship is not just a theoretical curiosity but forms the foundation for practical applications across numerous scientific and engineering disciplines. The ability to derive, visualize,

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