Use Spherical Coordinates. (a) Find The Volume Of The Solid That Lies Above The Cone = /3 And Below The

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Introduction

In the realm of multivariable calculus, calculating the volume of complex three-dimensional shapes often requires advanced coordinate systems. Spherical coordinates provide a powerful tool for solving problems involving spheres, cones, and other symmetric solids. This article explores the application of spherical coordinates to find the volume of a specific solid: the region lying above a cone defined by a particular angle and below another surface. We will systematically analyze the problem, convert the boundaries into spherical coordinates, set up the integral, and compute the volume.

Understanding the Problem

What is Given?

The problem states: "Find the volume of the solid that lies above the cone = /3 and below the..." (assuming the second boundary is a sphere or another surface). The key elements are:

- The cone is defined by an angular boundary, specifically, the angle $\theta = \pi/3$.
- The "above" and "below" refer to the spatial regions between the cone and some other surface, which, based on typical problems, could be a sphere or a plane.

Clarification of Boundaries

Given the partial statement, the common scenario in spherical coordinates problems is:

- The upper boundary: a sphere of radius R.
- The lower boundary: the cone given by $\theta = \pi/3$, which divides the space into a conical region.

The goal is to find the volume of the region:

- Above the cone $\theta = \pi/3$,
- Below the sphere $r = R$.

Assumptions for Complete Clarity

Since the original problem is incomplete, the typical complete problem involves:

> Find the volume of the solid lying above the cone $\theta = \pi/3$ and

below the sphere $r = R$.

This is a standard problem encountered in multivariable calculus, and we'll proceed with this interpretation.

Spherical Coordinates: An Overview

Definition and Transformation

Spherical coordinates (ρ, ϕ, θ) are related to Cartesian coordinates (x, y, z) by:

$$
\begin{cases}
x = \rho \sin \phi \cos \theta \\
y = \rho \sin \phi \sin \theta \\
z = \rho \cos \phi
\end{cases}
$$

where:

- $\rho \geq 0$ is the distance from the origin,
- $\phi \in [0, \pi]$ is the polar angle measured from the positive z-axis,
- $\theta \in [0, 2\pi]$ is the azimuthal angle in the xy-plane.

Volume Element in Spherical Coordinates

The differential volume element is:

$$
dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
$$

This expression is central to setting up integrals for volume calculations.

Setting Up the Surface Boundaries

The Sphere $r = R$

The sphere boundary in spherical coordinates:

$$
\rho = R
$$

The Cone $\theta = \pi/3$

In standard spherical coordinates, the cone boundary is often expressed in terms of ϕ or θ.

Important Note: In the context of spherical coordinates:

- ϕ (polar angle): angle from the positive z-axis,

- θ (azimuthal angle): angle in the xy-plane.

Given that the problem involves the cone $\theta = \pi/3$, this suggests the cone opens along the z-axis with an angular boundary in the azimuthal plane.

However, in many problems involving cones, the cone is defined by an angle from the z-axis, often written as $\phi = \alpha$.

Clarifying the Cone's Definition

Cone Defined by $\phi = \pi/3$

If the cone is symmetric about the z-axis, its boundary is given by:

$$
\phi = \alpha
$$

where α is the half-angle of the cone.

In this case:

- The region above the cone corresponds to $\phi \leq \pi/3$,
- The region below the cone corresponds to $\phi \geq \pi/3$.

Given the partial statement, and typical problem structures, it is more consistent that the cone is defined by $\phi = \pi/3$, rather than θ.

Final Assumption for the Problem

Assumption:

- The cone is defined by $\phi = \pi/3$,
- The sphere boundary is at $\rho = R$,
- The region of interest is above the cone (i.e., $\phi \leq \pi/3$),
- The volume is contained within the sphere ($0 \leq \rho \leq R$).

Computing the Volume

Step 1: Setting the Integration Limits

- θ: Since the problem involves a symmetric cone around the z-axis, θ spans the full azimuthal angle:

$$
0 \leq \theta \leq 2\pi
$$

- ϕ: For the region above the cone, the polar angle varies from:

\[

0 \leq \phi \leq \pi/3
\]

- \(\rho\): From the origin out to the sphere:

\[
0 \leq \rho \leq R
\]

Step 2: Setting Up the Triple Integral

The volume \(V\) is:

\[
V = \int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{0}^{R} \rho^2 \sin \phi \, d\rho
\, d\phi \, d\theta
\]

Step 3: Computing the Integral

The integral separates into three parts:

\[
V = \left(\int_{0}^{2\pi} d\theta \right) \times \left(\int_{0}^{\pi/3}
\sin \phi \, d\phi \right) \times \left(\int_{0}^{R} \rho^2 \, d\rho \right)
\]

Calculate each:

1. Azimuthal integral:

\[
\int_{0}^{2\pi} d\theta = 2\pi
\]

2. Polar angle integral:

\[
\int_{0}^{\pi/3} \sin \phi \, d\phi = -\cos \phi \big|_{0}^{\pi/3} = -\cos
(\pi/3) + \cos 0 = -\frac{1}{2} + 1 = \frac{1}{2}
\]

3. Radial integral:

\[
\int_{0}^{R} \rho^2 \, d\rho = \frac{\rho^3}{3} \big|_{0}^{R} = \frac{R^3}{3}
\]

Step 4: Final Expression for Volume

Putting it all together:

\[

$$
V = 2\pi \times \frac{1}{2} \times \frac{R^3}{3} = \pi \times \frac{R^3}{3} = \frac{\pi R^3}{3}
$$

Summary of Results

- The volume of the solid above the cone $\phi = \pi/3$ and below the sphere $r = R$ is:

$$
\boxed{V = \frac{\pi R^3}{3}}
$$

- This result applies when the cone is defined by $\phi = \pi/3$ (i.e., a cone opening downward from the z-axis).

Additional Considerations

Variations in the Boundary Conditions

- If the boundary is a different surface, such as a paraboloid or an oblique plane, the integral limits and the coordinate transformation would change accordingly.
- For cones defined by $\theta = \text{constant}$, the problem reduces to an angular wedge, and the limits in θ would be from the cone's azimuthal angle to another boundary.
- For cones symmetric about the z-axis, the most natural boundary is often expressed in terms of ϕ.

Application of Spherical Coordinates in Real-World Problems

- Astrophysics: Calculating volumes of celestial bodies or regions of space.
- Engineering: Analyzing the volume of conical tanks or regions with spherical symmetry.
- Mathematics: Solving integrals involving spheres and cones in multivariable calculus courses.

Conclusion

Using spherical coordinates simplifies the process of calculating volumes bounded by spheres and cones due to the symmetry of these shapes. By converting the boundaries into the spherical coordinate system, setting appropriate limits, and integrating over the defined region, we obtain precise volume measurements efficiently. The key steps involve understanding the geometric boundaries,

Frequently Asked Questions

How do I set up the triple integral in spherical coordinates to find the volume of a solid above the cone $\theta = \pi/3$?

To set up the triple integral, express the volume as an integral in spherical coordinates (ρ, θ, φ), where θ is the azimuthal angle, φ is the polar angle, and ρ is the radius. Since the cone is given by $\theta = \pi/3$, the region above the cone corresponds to $\theta \geq \pi/3$. The limits are ρ from 0 to the boundary surface (often given or derived), φ from 0 to $\pi/2$ (assuming the solid is above the xy-plane), and θ from $\pi/3$ to some maximum (e.g., 2π), depending on the symmetry. The volume element is $\rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$.

What is the significance of the cone $\theta = \pi/3$ in spherical coordinates, and how does it influence the volume calculation?

The cone $\theta = \pi/3$ represents a boundary where the polar angle φ (measured from the positive z-axis) determines the cone's surface. In the context of spherical coordinates, this boundary restricts the angular domain of integration. When calculating volume above this cone, you integrate over all points with $\theta \geq \pi/3$, effectively limiting the solid to the region outside (or inside) this cone, which directly influences the limits of the angular variable in the integral.

How can I determine the limits for ρ when calculating the volume above the cone in spherical coordinates?

The limits for ρ depend on the upper boundary surface of the solid. If the solid extends up to a sphere of radius R, then ρ varies from 0 to R for each point. If the boundary surface is a different surface (e.g., $z = k$ in Cartesian coordinates), you need to express that surface in spherical coordinates and solve for ρ to find its maximum value at each (θ, φ). If no upper boundary is specified, you may consider an integral up to a certain radius R or infinity for unbounded regions.

How do I convert the cone equation from Cartesian to spherical coordinates for volume calculations?

A cone with equation $z = r \tan \theta_0$ (or similar) in Cartesian coordinates can be converted by expressing z, r, and θ in spherical coordinates: $z = \rho \cos \varphi$, $r = \rho \sin \varphi$, and θ remains the same. The cone $\theta = \pi/3$ corresponds to a boundary where the polar angle $\varphi = \pi/3$, because in spherical coordinates, the surface of the cone is given by $\varphi = \pi/3$. This allows you to set the angular bounds directly in spherical coordinates.

What is the formula for the volume element in spherical coordinates, and how is it used in the integral?

The volume element in spherical coordinates is $dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$. When setting up the integral for volume, you multiply the integrand (which is 1 for volume) by this element, integrating over the specified limits for ρ, φ, and θ. This accounts for the differential volume in spherical coordinates and ensures correct calculation of the total volume.

Can you provide a step-by-step method to compute the volume of the solid above the cone θ = π/3?

Yes. Step 1: Identify the boundary surface and express it in spherical coordinates. Step 2: Set up the triple integral with appropriate limits: ρ from 0 to the boundary surface, φ from 0 to the upper limit (often π/2), and θ from π/3 to the full rotation (e.g., 2π). Step 3: Write the volume integral as $\iiint \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$ over these limits. Step 4: Evaluate the integral in order, simplifying at each step. Step 5: Interpret the result as the volume of the solid above the cone.

How does symmetry simplify the process of calculating the volume above the cone in spherical coordinates?

Symmetry, such as rotational symmetry about the z-axis, allows you to integrate over a full angular range (like θ from 0 to 2π) without concern for irregularities. This simplifies the limits and reduces complexity, enabling straightforward evaluation of the integral. Additionally, symmetry can sometimes reduce the integral to a single variable if the problem allows, making calculations more manageable.

Are there common pitfalls to watch out for when using spherical coordinates to find volume above a cone?

Yes. Common pitfalls include incorrect limits for the angular variables, misrepresenting the boundary surfaces in spherical coordinates, overlooking the Jacobian (the $\rho^2 \sin\varphi$ factor), and confusing the angles (θ vs. φ). It's also important to verify the domain boundaries carefully and ensure the limits correspond correctly to the physical region described in the problem.

Additional Resources

Use Spherical Coordinates. (a) Find The Volume Of The Solid That Lies Above The Cone = π/3 And Below The

When tackling three-dimensional volume problems, especially those involving regions bounded by curved surfaces like cones and spheres, using spherical coordinates can significantly simplify the process. Spherical coordinates transform complex boundary surfaces into more manageable equations, making the calculation of volumes more straightforward. In this guide, we will explore how to use spherical coordinates to find the volume of a solid lying above a specific cone—namely, the cone defined by an angle of π/3—and below a sphere or another boundary surface. This approach is a classic example of how coordinate transformations can streamline intricate volume integrals.

Understanding Spherical Coordinates

Before diving into the specific problem, it’s essential to understand the fundamentals of spherical coordinates. Unlike Cartesian coordinates (x, y, z), spherical coordinates express points in three-dimensional space using a radius and two angles:

- r (radius): The distance from the origin to the point.

- θ (theta): The angle in the xy-plane from the positive x-axis, typically ranging from 0 to 2π.
- φ (phi): The angle measured from the positive z-axis down toward the xy-plane, ranging from 0 to π.

Conversion formulas:

- $x = r \sin \varphi \cos \theta$
- $y = r \sin \varphi \sin \theta$
- $z = r \cos \varphi$

Volume element in spherical coordinates:

The volume element dV in spherical coordinates is:

$$dV = r^2 \sin \varphi \, dr \, d\varphi \, d\theta$$

This form simplifies many volume calculations involving spheres, cones, and other curved surfaces.

Setting Up the Problem

Suppose we are asked to find the volume of the solid that lies above the cone and below a sphere or another boundary surface. The problem statement might read:

"Find the volume of the solid that lies above the cone defined by $\theta = \pi/3$ and below the sphere $r = R$."

For the purpose of this guide, we'll assume the boundary surfaces are:

- Above the cone: the cone is defined by an angle $\theta = \pi/3$.
- Below the sphere: the sphere is defined by $r = R$ (a constant radius).

Note: In some problems, the cone might be described differently, such as by an equation involving φ, but for this example, we'll proceed with the cone specified by an angular boundary in θ.

Visualizing the Region

Visualizing the region is crucial:

- The cone $\theta = \pi/3$ divides the space into two regions; the region above the cone corresponds to θ ranging from 0 to $\pi/3$.
- The sphere $r = R$ caps the region at a fixed radius, so r ranges from 0 to R.
- The azimuthal angle θ varies from 0 to $\pi/3$ (or possibly 0 to 2π if the cone is symmetric around an axis).

Depending on the problem's specifics, the limits could vary. For simplicity, let's assume:

- θ varies from 0 to $\pi/3$.
- φ varies from 0 to π.
- r varies from 0 to R.

Step-by-Step Guide to Find the Volume

1. Identify the Boundaries

- Radial boundary: from 0 to R (the sphere boundary).
- Angular boundary in φ: typically from 0 to π, covering the entire sphere vertically.
- Angular boundary in θ: from 0 to $\pi/3$, representing the cone's boundary.

2. Set Up the Integral

Using spherical coordinates, the volume V is:

$V = \iiint r^2 \sin \varphi \, dr \, d\varphi \, d\theta$

The limits are:

- r: 0 to R
- φ: 0 to π
- θ: 0 to $\pi/3$

Thus,

$V = \int_0^{\pi/3} \int_0^{\pi} \int_0^{R} r^2 \sin \varphi \, dr \, d\varphi \, d\theta$

3. Integrate with Respect to r

$\int_0^{R} r^2 \, dr = (1/3) R^3$

Now, the volume integral becomes:

$V = (1/3) R^3 \int_0^{\pi/3} \int_0^{\pi} \sin \varphi \, d\varphi \, d\theta$

4. Integrate with Respect to φ

$\int_0^{\pi} \sin \varphi \, d\varphi = 2$

Therefore,

$V = (1/3) R^3 \times 2 \times \int_0^{\pi/3} d\theta$

5. Integrate with Respect to θ

$\int_0^{\pi/3} d\theta = \pi/3$

Putting it all together:

$V = (1/3) R^3 \times 2 \times (\pi/3) = (2/3) R^3 \times (\pi/3) = (2\pi R^3) / 9$

Final Result

The volume of the solid lying above the cone $\theta = \pi/3$ and below the sphere $r = R$ is:

$V = (2\pi R^3) / 9$

Extending the Analysis: More Complex Boundaries

While the above example considers a straightforward boundary setup, real-world problems often involve more complex regions. Here are some tips for handling such cases:

Handling Boundaries Defined by ψ

If the cone is defined by an angle $\psi = \psi_0$, the limits change:

- ψ varies from 0 to ψ_0 (the cone's surface).
- θ covers the full circle (0 to 2π).

In this case:

$$V = \int_0^{2\pi} \int_0^{\psi_0} \int_0^{R} r^2 \sin \psi \, dr \, d\psi \, d\theta$$

The integrated result would involve:

$$\int_0^{\psi_0} \sin \psi \, d\psi = 1 - \cos \psi_0$$

which adjusts the volume accordingly.

Handling Multiple Regions

Sometimes the region is split into multiple parts, requiring piecewise integration or subtracting volumes of known shapes to find the desired volume.

Practical Applications of Using Spherical Coordinates

- Astrophysics: Calculating volumes of celestial bodies or regions of space.
- Engineering: Determining the volume of parts with curved boundaries.
- Mathematics: Solving triple integrals involving spheres and cones.

By mastering use spherical coordinates, you can efficiently handle a wide range of three-dimensional volume problems that would be cumbersome in Cartesian coordinates.

Summary

- Use spherical coordinates to simplify volume calculations involving spheres, cones, and other curved surfaces.
- Carefully identify the boundaries in terms of r, ψ, and θ.
- Set up the triple integral with the correct limits based on the geometric region.
- Perform the integrations step-by-step, starting with r, then ψ, then θ.
- Always visualize the region to ensure the limits are correctly assigned.

This approach transforms complex boundary surfaces into manageable integrals, enabling precise calculation of volumes for a variety of geometric regions. Whether you are solving academic problems or tackling real-world applications, using spherical coordinates is a powerful tool in your mathematical toolkit.

Remember: Practice with different boundary conditions and regions to build intuition and confidence in using spherical coordinates for volume calculations.

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