3 Let E Be The Solid That Lies Under The Plane Z = 4x + Y And Above The Region In The Xy- Plane Enclosed

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Understanding the geometry of solids in three-dimensional space is fundamental in calculus, physics, engineering, and various applied sciences. Among the many intriguing problems in multivariable calculus is determining the volume of a solid bounded by a surface and a region in the XY-plane. In this article, we will delve into the problem of finding the volume of the solid E, which lies beneath the plane $z = 4x + y$ and above a specific region in the XY-plane.

This problem exemplifies the application of double integrals, a powerful technique for calculating volumes and other quantities in multivariable calculus. By carefully analyzing the region in the XY-plane and the surface that bounds the solid, we can set up and evaluate the appropriate integrals. This approach not only provides the volume but also enhances understanding of how geometric intuition translates into integral calculus.

Understanding the Geometric Setup

Before diving into calculations, it is crucial to comprehend the geometric configuration of the problem.

The Plane $z = 4x + y$

The surface $z = 4x + y$ is a plane in three-dimensional space. It has a linear form, combining variables x and y, with the coefficients 4 and 1, respectively. The plane extends infinitely in all directions unless bounded by a specific region in the XY-plane.

Key characteristics of this plane include:

- Intercepts:
- When $x = 0$, $z = y$.
- When $y = 0$, $z = 4x$.
- Orientation:
- The plane tilts depending on the coefficients, with a steeper slope in the x-direction due to the coefficient 4.

Understanding the plane's orientation helps visualize the shape of the solid E.

The Region in the XY-plane

The problem states that the solid E lies "above the region in the XY-plane enclosed," but the specific region is not explicitly described in the initial statement. Typically, such problems specify a region $R \subset \mathbb{R}^2$, which could be:

- A simple geometric shape like a rectangle, triangle, or circle.
- A region bounded by lines, curves, or other polygons.

For the purpose of this discussion, let's assume a common and illustrative case:

Assumption: The region R is a rectangle defined by

$$
a \leq x \leq b, \quad c \leq y \leq d
$$

where a, b, c, d are real numbers with $a < b$ and $c < d$.

This assumption simplifies the setup while illustrating the key steps involved in solving such problems.

Setting Up the Double Integral for the Volume

The volume V of the solid E is obtained by integrating the height $z = 4x + y$ over the region R in the XY-plane:

$$
V = \iint_{R} (\text{top surface} - \text{bottom surface}) \, dA
$$

Since the solid is bounded below by the XY-plane (where $z = 0$) and above by the plane $z = 4x + y$, the integral simplifies to:

$$
V = \iint_{R} (4x + y) \, dA
$$

assuming the region R is where $4x + y \geq 0$. If the region includes points where $4x + y < 0$, the integral should be adjusted accordingly, but for simplicity, we'll proceed with the assumption that the region is within the positive domain or is defined such that the integrand remains non-negative.

Expressing the Double Integral

In rectangular coordinates, the double integral over the rectangle $R = [a, b] \times [c, d]$ is:

$$
V = \int_{x=a}^{b} \int_{y=c}^{d} (4x + y) \, dy \, dx
$$

Alternatively, if the region R has a more complex shape, the limits of integration need to be adjusted accordingly, potentially involving variable limits or changing the order of integration.

Calculating the Volume: Step-by-Step Process

Let's now perform the calculation assuming the rectangular region R.

Step 1: Integrate with respect to y

$$
V = \int_{a}^{b} \left[\int_{c}^{d} (4x + y) \, dy \right] dx
$$

Inner integral:

$$
\int_{c}^{d} (4x + y) \, dy = \int_{c}^{d} 4x \, dy + \int_{c}^{d} y \, dy
$$

Since $4x$ is constant with respect to y:

$$
= 4x (d - c) + \left[\frac{y^2}{2} \right]_{c}^{d} = 4x (d - c) + \frac{d^2 - c^2}{2}
$$

Step 2: Integrate with respect to x

Now, the volume becomes:

$$
V = \int_{a}^{b} \left(4x (d - c) + \frac{d^2 - c^2}{2} \right) dx
$$

Break into two integrals:

$$
V = (d - c) \int_{a}^{b} 4x \, dx + \frac{d^2 - c^2}{2} \int_{a}^{b} dx
$$

Calculate each:

- First integral:

$$
4 (d - c) \left[\frac{x^2}{2} \right]_{a}^{b} = 4 (d - c) \left(\frac{b^2 - a^2}{2} \right) = 2 (d - c) (b^2 - a^2)
$$

- Second integral:

$$
\frac{d^2 - c^2}{2} (b - a)
$$

Putting it all together:

$$
V = 2 (d - c) (b^2 - a^2) + \frac{d^2 - c^2}{2} (b - a)
$$

This formula provides the volume of the solid E over the rectangular region $R = [a, b] \times [c, d]$.

Generalization to Other Regions

While the previous calculation assumes a rectangular region, real-world problems often involve more complex regions.

Examples of Enclosed Regions

- Triangular Region:

For example, the region bounded by $y = 0$, $y = x$, and x between a and b.

- Circular Region:

For example, the disk $x^2 + y^2 \leq r^2$.

- Polygonal Regions:

Any polygonal region can be integrated over by dividing into simpler subregions.

Adjusting Limits for Complex Regions

In such cases, the limits of integration depend on the region's equations:

- Iterated integrals can be set up with either $dy \, dx$ or $dx \, dy$, choosing the order that simplifies the limits.

- Change of variables (e.g., polar coordinates) can be employed for regions with symmetry or circular boundaries.

Visualizing the Solid and the Region

Visualization is crucial for understanding the problem:

- 3D Graphs: Plotting the plane $z = 4x + y$ along with the XY-region helps perceive the shape and volume.

- Cross-Sections: Analyzing horizontal or vertical slices provides insight into the volume's accumulation.

- Software Tools: Utilizing graphing software or CAD tools enhances comprehension.

Applications of the Solid E and Its Volume

Understanding the volume of such solids has practical implications:

- Engineering: Calculating material quantities in manufacturing.

- Physics: Determining mass, center of mass, or moments of inertia for objects with variable density.

- Environmental Science: Estimating volumes of terrain features or pollutant distributions.

- Economics & Data Science: Modeling multi-variable functions over specific regions.

Conclusion

The problem of finding the volume of the solid E that lies beneath the plane $z = 4x + y$ and above a region in the XY-plane encapsulates core concepts of multivariable calculus, including setting up double integrals, understanding regions of integration, and geometric visualization.

By carefully analyzing the boundary surfaces and the region, formulating the integral limits, and executing step-by-step calculations, we can accurately determine the volume. Although the specific region was assumed to be rectangular for simplicity

Frequently Asked Questions

What is the geometric description of the solid E in the problem?

The solid E is the region in space lying beneath the plane $z = 4x + y$ and above the enclosed region R in the xy-plane.

How do you determine the boundary region R in the xy-plane for the solid E?

Region R is determined by the projection of the solid onto the xy-plane, which is the region enclosed by the projection of the plane's boundary, typically given by the intersection with other surfaces or specified limits.

What is the method to compute the volume of the solid E?

The volume can be found by setting up a double integral over region R of the function $z = 4x + y$, i.e., $V = \iint_R (4x + y)\, dA$.

How can the order of integration be chosen when calculating the volume?

The order of integration depends on the shape of region R; typically, one chooses the order that simplifies the limits, such as integrating with respect to y first and then x, or vice versa, based on the boundary equations.

What are common techniques for describing the region R for such a problem?

Region R can often be described using inequalities derived from the boundary curves or lines in the xy-plane, such as inequalities involving x and y, or using coordinate transformations if the region has a more complex shape.

Can the volume of the solid E be interpreted as the integral of the plane over the region R?

Yes, the volume under the plane $z = 4x + y$ over the region R is obtained by integrating the function $4x + y$ over R, effectively summing the infinitesimal areas multiplied by the height of the plane.

Additional Resources

Let E Be The Solid That Lies Under The Plane Z = 4x + y And Above The Region In The XY-Plane Enclosed is a classic problem in multivariable calculus that offers a rich exploration of solid regions, volume calculations, and the application of double integrals. This problem encapsulates fundamental concepts such as setting up the bounds of integration, visualizing three-dimensional regions, and understanding the geometric relationships between planes and regions in the xy-plane. It is an essential exercise for students and practitioners aiming to deepen their understanding of spatial reasoning and integral calculus.

In this comprehensive review, we will analyze the problem from multiple perspectives, including its geometric interpretation, the methods for calculating its volume, the significance of the boundary region, and various approaches to solving it. By dissecting each element, we aim to provide a detailed understanding of how to approach similar problems and the key features, advantages, and challenges associated with this type of calculus problem.

Understanding the Geometric Setup

The Plane Z = 4x + y

The plane described by the equation $Z = 4x + y$ is a linear surface in three-dimensional space. Its slope and orientation are determined by the coefficients of x and y. Specifically:

- The plane increases in height (Z) as x increases, scaled by a factor of 4.
- The height also depends linearly on y, with a coefficient of 1.
- The plane intersects the xy-plane where $Z=0$, which occurs along the line $y = -4x$.

Understanding the orientation of this plane is crucial because it influences how the solid E is bounded and how the volume calculation will be approached.

The Enclosed Region in the XY-Plane

While the problem mentions that the solid E lies above a certain region in the xy-plane, it does not specify that region explicitly in the prompt. Typically, such problems specify an enclosed region R in the xy-plane, over which the double integral is computed. The shape of R—whether it is rectangular, circular, or bounded by lines—affects both the setup and the complexity of the integral.

Commonly, the region R might be a polygon, such as a triangle or rectangle, or a more complex shape. The key is that R must be clearly defined by inequalities, such as:

- Rectangular: $a \leq x \leq b$, $c \leq y \leq d$
- Circular: $x^2 + y^2 \leq r^2$
- Polygonal: defined by boundary lines

For the purpose of this review, we'll assume R is a well-defined, bounded region in the xy-plane.

Visualizing the Solid E

Graphical Representation

Visualizing the solid E involves understanding its boundaries both in the xy-plane and along the z-axis:

- The bottom boundary is the region R in the xy-plane.
- The top boundary is the plane $Z = 4x + y$.

Imagine the region R lying flat on the xy-plane. The surface above it rises linearly according to the plane equation. The solid E is thus a 'skewed' shape whose top surface is the plane, and the base is the region R.

A useful approach is to sketch the plane and the region R in three dimensions to see how the solid is formed. Such visualization aids in setting up the correct limits of integration.

Implications of the Plane's Orientation

The plane's inclination—steep in the x-direction and moderate in the y-direction—implies that the height of the solid varies significantly depending on the position within R. For example:

- For positive x-values, the height increases as x increases.
- For y-values, the height increases with y, but only linearly.

This variation means that the volume calculation must carefully account for the changing height across the region R.

Calculating the Volume of E

Setting Up the Double Integral

The volume of the solid E can be computed using a double integral over the region R:

$$ V = \iint_{R} (4x + y) \, dA $$

This is because the height of the solid at each point (x, y) is given by $Z = 4x + y$, and the area element dA can be expressed in terms of dx and dy.

The key steps involve:

1. Defining the region R precisely via inequalities.
2. Choosing an order of integration (dx dy or dy dx).
3. Setting the limits of integration based on R's boundaries.

For example, if R is a rectangle with sides $a \leq x \leq b$ and $c \leq y \leq d$, then:

$$ V = \int_{x=a}^{b} \int_{y=c}^{d} (4x + y) \, dy \, dx $$

Alternatively, if R is more complex, the limits might depend on each other, requiring careful setup.

Solving the Integral

Once the limits are established, the integration proceeds as follows:

- Integrate with respect to y (or x), treating the other as constant.
- Then integrate the resulting expression with respect to x (or y).

For instance, if R is rectangular:

$$ V = \int_{a}^{b} \left[\int_{c}^{d} (4x + y) \, dy \right] dx $$

which simplifies to:

$$ V = \int_{a}^{b} \left[4x y + \frac{1}{2} y^2 \right]_{c}^{d} dx $$

$$ V = \int_{a}^{b} \left(4x (d - c) + \frac{1}{2} (d^2 - c^2) \right) dx $$

and then integrate with respect to x.

Pros:

- Straightforward for rectangular regions.
- Clear geometric interpretation.

Cons:

- More complex regions require careful boundary determination.
- Changing the order of integration may be necessary for easier computation.

Alternative Coordinate Systems

In some cases, transforming to other coordinate systems simplifies the integral:

- Polar coordinates: Useful if R is circular or radial.
- Change of variables: Can simplify the region R, especially if boundaries are linear or quadratic.

However, for the plane Z = 4x + y, Cartesian coordinates are often most straightforward.

Features and Significance of the Problem

Features

- Linear boundary surface: The plane Z = 4x + y is a simple, linear surface, making the integral manageable.
- Enclosed region R: The shape of R dictates the complexity of the volume calculation.
- Multiple methods of solution: Depending on R, the problem can be approached through direct integration, substitution, or coordinate transformations.

Significance in Teaching and Practice

- Fundamental application of double integrals: This problem demonstrates how to compute volume via integration over a projected region.
- Visualization skills: Encourages students to visualize 3D regions and surfaces.
- Understanding geometric boundaries: Highlights the importance of boundary determination in multivariable calculus.
- Application scope: Relevant in physics, engineering, and other sciences where volume under a surface is required.

Pros and Cons of Approaches

Pros:

- Direct double integration provides clarity and straightforwardness for simple regions.
- Coordinate transformations can simplify complex regions.
- Visualization enhances understanding of the region and the volume.

Cons:

- For irregular regions, setting up limits can be complicated.
- Transformations may require additional algebraic work.
- Misinterpretation of bounds can lead to errors.

Conclusion and Final Remarks

The problem of determining the volume of the solid E lying under the plane Z = 4x + y and above a region R in the xy-plane is a fundamental exercise in multivariable calculus. It encapsulates core concepts such as setting up double integrals, understanding the geometric relationships between surfaces and regions, and applying integration techniques effectively.

By carefully analyzing the shape of the boundary region, choosing suitable coordinates, and executing the integration step-by-step, one can accurately compute the volume. The problem also underscores the importance of visualization, boundary determination, and the strategic choice of integration order.

In practice, such problems serve as excellent pedagogical tools, reinforcing theoretical knowledge with practical computation and geometric intuition. Whether for academic purposes or real-world applications, mastering this type of problem is essential for anyone working with multivariable calculus or related fields.

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