6. Write An Iterated Integral That Gives The Volume Of The Solid Bounded By The Surface F(x, Y)=x Y Over

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Understanding how to determine the volume of a solid bounded by a surface is a fundamental concept in multivariable calculus. In particular, when the surface is given by a function such as $F(x, y) = xy$, forming an iterated integral allows us to compute the volume beneath this surface over a specified region in the xy-plane. This process involves carefully defining the region of integration and setting up the integral in a way that aligns with the geometric boundaries of the solid. This article will guide you through the steps to write the appropriate iterated integral for the volume of such a solid, exploring various scenarios and providing examples to solidify your understanding.

Understanding the Surface $F(x, y) = xy$

Before delving into the integral setup, it's essential to understand the nature of the surface $z = xy$:

- Surface Characteristics:
The surface $z = xy$ is a saddle-shaped surface, often referred to as a hyperbolic paraboloid. It extends infinitely in all directions unless bounded by specific limits.

- Symmetry:
The surface exhibits symmetry with respect to the origin and has different behaviors in different quadrants:
- Quadrants I and III: $xy > 0$ (positive values)
- Quadrants II and IV: $xy < 0$ (negative values)

- Boundaries of the Solid:
Since the problem involves a bounded solid, the region over which we integrate in the xy-plane must be finite and explicitly defined.

Defining the Region of Integration R

The first crucial step in setting up an iterated integral is identifying the region R in the xy-plane over which the volume is computed. Depending on the problem's specifics, R

could be:

- A rectangular region
- A circular or elliptical region
- A region bounded by curves

Common types of regions include:

1. Rectangular Region $a \leq x \leq b$, $c \leq y \leq d$
- Simplifies the setup as the limits are constant.

2. Region bounded by curves, such as $y = g_1(x)$ and $y = g_2(x)$ over $a \leq x \leq b$
- Often used when the boundary curves are more complex.

3. Circular or Polar Regions
- When the region is circular, using polar coordinates can simplify the integral.

Example of Region Definition:

Suppose the solid is bounded above by the surface $z = xy$ and below by the xy-plane (i.e., $z=0$), over a region R in the xy-plane.

- If the boundary is a rectangle: $a \leq x \leq b$, $c \leq y \leq d$.
- If the boundary is a circle of radius r: $x^2 + y^2 \leq r^2$.

The choice of region impacts the limits of integration.

Formulating the Volume as an Iterated Integral

The volume V of the solid bounded by the surface $z = xy$ over the region R in the xy-plane is given by:

$$
V = \iint_{R} xy \, dA
$$

Where dA is the differential area element in the xy-plane, which can be expressed as:

- $dx\,dy$ if integrating with respect to x first,
- $dy\,dx$ if integrating with respect to y first.

The specific order of integration depends on the shape of R and which limits are simpler to express.

General Steps to Set Up the Iterated Integral:

1. Identify the region R and determine the limits for x and y.

2. Choose the order of integration (either $dx\,dy$ or $dy\,dx$) based on the region's shape.

3. Write the integral:

$$
V = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} xy \, dy \, dx
$$

or

$$
V = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} xy \, dx \, dy
$$

Examples of Setting Up the Iterated Integral

Example 1: Rectangular Region

Suppose the region R is a rectangle defined by:

$$
0 \leq x \leq 2, \quad 0 \leq y \leq 3
$$

The volume is:

$$
V = \int_{x=0}^{2} \int_{y=0}^{3} xy \, dy \, dx
$$

Calculating the integral:

$$
V = \int_{0}^{2} \left(\int_{0}^{3} xy \, dy \right) dx
= \int_{0}^{2} x \left(\int_{0}^{3} y \, dy \right) dx
= \int_{0}^{2} x \left[\frac{y^2}{2} \right]_{0}^{3} dx
= \int_{0}^{2} x \left(\frac{9}{2} \right) dx
= \frac{9}{2} \int_{0}^{2} x \, dx
= \frac{9}{2} \left[\frac{x^2}{2} \right]_0^{2}
= \frac{9}{2} \times \frac{4}{2}
= \frac{9}{2} \times 2
= 9
$$

Result: The volume under $z=xy$ over this rectangular region is 9.

Example 2: Region bounded by curves

Suppose R is bounded by:

$$
x \text{ from } 0 \text{ to } 1, \quad y \text{ from } 0 \text{ to } y=x
$$

The region is a triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$.

The volume:

$$
V = \int_{x=0}^{1} \int_{y=0}^{y=x} xy \, dy \, dx
$$

Calculations:

$$
V = \int_{0}^{1} x \left(\int_{0}^{x} y \, dy \right) dx
= \int_{0}^{1} x \left[\frac{y^2}{2} \right]_0^{x} dx
= \int_{0}^{1} x \times \frac{x^2}{2} dx
= \frac{1}{2} \int_{0}^{1} x^3 \, dx
= \frac{1}{2} \left[\frac{x^4}{4} \right]_0^{1}
= \frac{1}{2} \times \frac{1}{4}
= \frac{1}{8}
$$

Result: The volume in this case is $1/8$.

Changing to Polar Coordinates for Circular Regions

When the region R is circular, it often simplifies calculations to convert to polar coordinates:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

and

$$
dA = r \, dr \, d\theta
$$

The surface $z = xy$ becomes:

$$
z = r^2 \cos \theta \sin \theta
$$

The volume integral over a circular region of radius R:

$$
V = \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{r=R} r^2 \cos \theta \sin \theta \times r \, dr \, d\theta
= \int_{\theta=\theta_1}^{\theta_2} \int_{0}^{R} r^3 \cos \theta \sin \theta \, dr \, d\theta
$$

This approach is especially useful when the region is symmetric about the origin or bounded by circles.

Special Considerations and Tips

- Choosing the Order of Integration:
Always select the order that simplifies the limits. If the region is easier to describe with x as a function of y, then integrate with respect to x first, and vice versa.

- Setting Correct Limits:
Carefully determine whether the limits are constants or functions, based on the region's boundary curves.

- Handling Negative Values:
Since $z=xy$ can be negative, the total volume (if considering the "unsigned" volume under the surface) might involve taking the absolute value or splitting the integral over positive and negative regions.

- Using Symmetry:
If the region or the function exhibits symmetry, exploit this to reduce

Frequently Asked Questions

What is the general approach to setting up an iterated integral for the volume of a solid bounded by the surface f(x, y) = xy?

The general approach involves identifying the projection of the solid onto the xy-plane and then integrating the function xy over that region, setting up the iterated integral as $\iint_D xy \, dy \, dx$, where D is the projection region.

How do you determine the limits of integration when setting up the iterated integral for the volume under f(x, y) = xy?

The limits are determined by analyzing the boundary surfaces and the projection region D in the xy-plane. Typically, you specify the bounds for x and y based on the region's shape and the given surface, ensuring the integral covers the entire volume.

Can you provide an example of an iterated integral for the volume of the solid bounded by f(x, y) = xy over a specific region?

Yes. For example, if the region D is a square with $0 \leq x \leq 1$ and $0 \leq y \leq 1$, the volume is given by $\int_0^1 \int_0^1 xy \, dy \, dx$.

How does the symmetry of the function f(x, y) = xy influence the setup of the integral?

Since xy is symmetric with respect to the axes, the integral setup can be simplified by exploiting symmetry, especially if the region D is symmetric about axes, potentially reducing computation effort.

What are common mistakes to avoid when writing an iterated integral for this volume problem?

Common mistakes include incorrect determination of the projection region D, mixing up the order of integration, and not properly setting the limits based on the boundary surfaces and the region's shape.

How does changing the order of integration affect setting up the integral for the volume under f(x, y) = xy?

Changing the order of integration requires re-evaluating the limits based on the region D. Sometimes, switching the order simplifies the integral, especially if one variable's limits are easier to express first.

If the region D is bounded by $x = 0$, $y = 0$, and $y = x$, how would you write the iterated integral for the volume?

The integral can be written as $\int_0^1 \int_0^x xy \, dy \, dx$, integrating with respect to y first, then x.

How would you interpret the physical meaning of the iterated integral $\int\int_D xy \, dy \, dx$ in terms of volume?

This integral computes the volume of the solid by summing up infinitesimal slices where each slice's height is given by the value of the surface xy over the region D in the xy-plane.

Are there specific tools or software that can assist in setting up and evaluating such iterated integrals?

Yes, tools like Wolfram Alpha, MATLAB, Mathematica, and Desmos can help set up, visualize, and evaluate iterated integrals for volume calculations, making the process more efficient and less error-prone.

Additional Resources

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Introduction

The task of determining the volume of a solid bounded by a surface is fundamental in multivariable calculus, providing insights into the spatial relationships and the behavior of functions in three dimensions. Specifically, when dealing with a surface defined by a function such as $F(x, y) = xy$, the process involves translating the geometric problem into an integral form that can be evaluated either analytically or numerically. The concept of iterated integrals becomes central to this endeavor, allowing us to systematically compute the volume enclosed between the surface and a specified region in the xy-plane.

In this comprehensive review, we will explore the methodology for constructing an iterated integral that computes the volume of the solid bounded by the surface $z = xy$. We will examine how to identify the appropriate projection region, set up the integral carefully, and interpret the results within the broader context of multivariable calculus. The discussion will be structured into key sections, providing clarity, depth, and analytical rigor.

Understanding the Geometric Context

The Surface $z = xy$

The surface $z = xy$ is a classic example in multivariable calculus, often referred to as a hyperbolic paraboloid. It exhibits saddle-shaped behavior, curving upward in some directions and downward in others.

Key properties:

- Symmetry: The surface is symmetric with respect to the origin, reflecting the fact that changing the signs of x and y affects z correspondingly.
- Sign behavior: For $x, y > 0$, the surface is positive; for $x > 0, y < 0$, negative; and similarly for other quadrants, leading to a saddle point at the origin.
- Unboundedness: The surface extends infinitely in all directions, implying that the volume calculation requires clear bounds in the xy-plane to produce a finite solid.

Bounded Region in the xy-plane

Since the surface extends infinitely, to compute a finite volume, we need to specify a bounded region R in the xy-plane over which the surface is "capped." The shape of this region directly influences the integral setup.

Common choices include:

- Rectangular regions, e.g., $a \leq x \leq b$, $c \leq y \leq d$.
- Circular or elliptical regions, requiring transformation into suitable coordinates.
- Regions defined by inequalities involving x and y.

In our discussion, we assume the region R is explicitly defined, as the integral's bounds depend on the problem's specific context.

Setting Up the Iterated Integral

General Strategy

To find the volume V of the solid bounded by the surface $z = xy$ over a region R, the fundamental approach involves integrating the "height" z over R:

$$
V = \iint_{R} xy \, dA
$$

where dA is the differential area element in the xy-plane.

Key steps:

1. Identify the region R: Expressed via inequalities in x and y.
2. Choose the order of integration: Decide whether to integrate with respect to y first, then x, or vice versa.
3. Set the bounds accordingly: Based on the shape of R and the order of integration.
4. Write the iterated integral: As a nested integral, e.g.,

$$
V = \int_{x=\text{lower}}^{x=\text{upper}}
\int_{y=\text{lower}(x)}^{y=\text{upper}(x)} xy \, dy \, dx
$$

or

$$
V = \int_{y=\text{lower}}^{y=\text{upper}}
\int_{x=\text{lower}(y)}^{x=\text{upper}(y)} xy \, dx \, dy
$$

Example: Rectangular Region

Suppose the bounded region R is a rectangle defined by:

$$
a \leq x \leq b, \quad c \leq y \leq d
$$

then the volume becomes:

$$
V = \int_{x=a}^{b} \int_{y=c}^{d} xy \, dy \, dx
$$

which can be evaluated step-by-step.

Constructing the Specific Iterated Integral

Step 1: Define the Region R

Let's consider a specific example where the region R is a rectangle, say:

$$
0 \leq x \leq a, \quad 0 \leq y \leq b
$$

This choice simplifies the bounds and allows for explicit computation.

Step 2: Write the Integral

Given the bounds are constant, the integral becomes:

$$
V = \int_{x=0}^{a} \int_{y=0}^{b} xy \, dy \, dx
$$

This form is straightforward and demonstrates the process clearly.

Step 3: Integrate with Respect to y

Compute the inner integral:

$$
\int_{0}^{b} xy \, dy = x \int_{0}^{b} y \, dy = x \left[\frac{y^2}{2} \right]_0^{b} = x \frac{b^2}{2}
$$

Step 4: Integrate with Respect to x

Now, integrate the resulting expression over x:

$$
V = \int_{0}^{a} x \frac{b^2}{2} \, dx = \frac{b^2}{2} \int_{0}^{a} x \, dx = \frac{b^2}{2} \left[\frac{x^2}{2} \right]_0^{a} = \frac{b^2}{2} \times \frac{a^2}{2} = \frac{a^2 b^2}{4}
$$

Result:

$$
\boxed{
V = \frac{a^2 b^2}{4}
}
$$

This explicit calculation exemplifies how the iterated integral directly yields the volume.

Generalization to Arbitrary Regions

Regions with Non-Constant Bounds

In more complex scenarios, the bounds for y may depend on x, or vice versa. For example, suppose the region R is bounded by:

$$
x \text{-bounds}: \quad x_1 \leq x \leq x_2
$$
$$
y \text{-bounds}: \quad y_1(x) \leq y \leq y_2(x)
$$

The volume becomes:

$$
V = \int_{x=x_1}^{x_2} \int_{y=y_1(x)}^{y_2(x)} xy \, dy \, dx
$$

The process of setting up the integral then involves:

- Carefully determining the functions $y_1(x)$ and $y_2(x)$.
- Ensuring the bounds are correctly oriented to cover the entire region without overlaps or gaps.
- Deciding the order of integration based on the ease of integration or the nature of the bounds.

Example: Triangular Region

Suppose R is a triangle bounded by:

- $y = 0$
- $y = x$
- x from 0 to 1

The iterated integral with respect to y first:

$$
V = \int_{x=0}^{1} \int_{y=0}^{y=x} xy \, dy \, dx
$$

Calculations proceed as follows:

- Inner integral:

$$
\int_{0}^{x} xy \, dy = x \left[\frac{y^2}{2} \right]_0^{x} = x \times \frac{x^2}{2} = \frac{x^3}{2}
$$

- Outer integral:

$$
V = \int_{0}^{1} \frac{x^3}{2} \, dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^{1} = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}
$$

Thus, the volume of the solid under the surface $z = xy$ over this triangular region is $\frac{1}{8}$.

Analytical and Computational Considerations

Choosing the Order of Integration

Deciding whether to integrate with respect to x first or y first can significantly influence the complexity of the calculation. Factors to consider include:

- Simplicity of bounds: When bounds are easier to express or evaluate in one order.

- Integrand structure: When the integrand simplifies more readily in one variable.
- Symmetry: Exploiting symmetry properties of the region or the integrand.

In the context of \(z = xy \), the integrand's separability suggests that both orders are feasible, but the specific bounds often guide the choice.

Numerical Approaches

For regions with complicated boundaries or integrands that resist closed-form solutions, numerical methods such as Monte Carlo integration

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