A) Find Analytical Expressions For The Magnitude And Phase Of \[G(s)=\frac{s}{(s+1)(s+10)} \] [6 Marks]

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Understanding the analytical expressions for the magnitude and phase of a transfer function is fundamental in control systems and signal processing. These expressions help engineers analyze the frequency response of systems, determine stability margins, and design appropriate controllers. In this article, we will derive the magnitude and phase expressions of the given transfer function:

\[
G(s) = \frac{s}{(s+1)(s+10)}
\]

and interpret their significance in practical applications.

Understanding the Transfer Function \(G(s) \)

Before deriving the analytical expressions, it is important to understand the structure of the transfer function:

\[
G(s) = \frac{s}{(s+1)(s+10)}

\]

This is a rational function with a zero at the origin $s=0$ and poles at $s=-1$ and $s=-10$. The transfer function describes the system's behavior in the Laplace domain, where $s = j\omega$ for frequency response analysis.

Replacing s with $j\omega$ for Frequency Response

To analyze the magnitude and phase, we evaluate $G(s)$ along the imaginary axis:

$$
G(j\omega) = \frac{j\omega}{(j\omega + 1)(j\omega + 10)}
$$

This substitution allows us to examine how the system behaves across different frequencies ω.

Deriving the Magnitude of $G(j\omega)$

The magnitude of the transfer function at a given frequency ω is:

$$
|G(j\omega)| = \frac{|j\omega|}{|j\omega + 1| \cdot |j\omega + 10|}
$$

Let's analyze each component:

1. Magnitude of numerator:

$$
|j\omega| = \sqrt{(0)^2 + (\omega)^2} = |\omega|
$$

2. Magnitude of denominator:

$$
|j\omega + a| = \sqrt{(\text{Re}(j\omega + a))^2 + (\text{Im}(j\omega + a))^2}
$$

Since $j\omega + a = a + j\omega$, the magnitude is:

$$
|j\omega + a| = \sqrt{a^2 + \omega^2}
$$

Applying this to each term:

$$
|j\omega + 1| = \sqrt{1^2 + \omega^2} = \sqrt{1 + \omega^2}
$$

$$
|j\omega + 10| = \sqrt{10^2 + \omega^2} = \sqrt{100 + \omega^2}
$$

3. Final magnitude expression:

Putting it all together:

$$
|G(j\omega)| = \frac{\omega}{\sqrt{1 + \omega^2} \times \sqrt{100 + \omega^2}}
$$

which simplifies to:

$$
\boxed{
|G(j\omega)| = \frac{\omega}{\sqrt{(1 + \omega^2)(100 + \omega^2)}}
}
$$

This expression provides the magnitude of the transfer function at any frequency ω.

Deriving the Phase of $G(j\omega)$

The phase of the transfer function at frequency ω is the sum of the phases of numerator and denominator:

$$
\angle G(j\omega) = \angle j\omega - \left(\angle (j\omega + 1) + \angle (j\omega + 10) \right)
$$

1. Phase of numerator:

$$

\angle j\omega =

\begin{cases}

\frac{\pi}{2}, & \omega > 0 \\

-\frac{\pi}{2}, & \omega < 0

\end{cases}

\]

Since we are typically interested in positive frequencies:

\[

\angle j\omega = \frac{\pi}{2}

\]

2. Phase of denominator components:

For each term \(j\omega + a = a + j\omega\), the phase is:

\[

\angle (a + j\omega) = \arctan \left(\frac{\omega}{a} \right)

\]

Applying to each:

\[

\angle (j\omega + 1) = \arctan \left(\frac{\omega}{1} \right) = \arctan(\omega)

\]

\[

\angle (j\omega + 10) = \arctan \left(\frac{\omega}{10} \right)

\]

3. Final phase expression:

Putting it all together:

$$
\boxed{
\angle G(j\omega) = \frac{\pi}{2} - \left(\arctan(\omega) + \arctan \left(\frac{\omega}{10} \right) \right)
}
$$

This expression describes the phase shift introduced by the system at any frequency ω.

Interpreting the Magnitude and Phase Expressions

The derived expressions for magnitude and phase are essential for understanding the frequency response of $G(s)$:

- Magnitude response shows how the amplitude of the output varies with frequency, indicating the system’s filter characteristics.
- Phase response reveals how the phase of the output signal shifts relative to the input at different frequencies, critical in stability analysis and controller design.

Key Observations and Practical Implications

1. Low-Frequency Behavior ($\omega \to 0$):

- Magnitude:

$$
|G(j0)| = \frac{0}{\sqrt{1} \times \sqrt{100}} = 0
$$

indicating the system acts as a high-pass filter with zero gain at DC.

- Phase:

$$
\angle G(0) = \frac{\pi}{2} - (0 + 0) = \frac{\pi}{2}
$$

corresponding to a +90° phase shift.

2. High-Frequency Behavior ($\omega \to \infty$):

- Magnitude:

$$
|G(j\omega)| \sim \frac{\omega}{\omega \times \omega} = \frac{1}{\omega} \to 0
$$

- Phase:

$$
\angle G(\infty) \to \frac{\pi}{2} - \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = -\frac{\pi}{2}
$$

indicating a phase lag approaching -90° at high frequencies.

3. Resonance and Bandwidth:

The magnitude peaks at certain frequencies, which can be calculated by analyzing the maximum of $|G(j\omega)|$. This is useful in filter design to determine cutoff frequencies.

Conclusion

In this detailed analysis, we have derived the analytical expressions for both the magnitude and phase of the transfer function $G(s) = \frac{s}{(s+1)(s+10)}$. These expressions are:

$$
|G(j\omega)| = \frac{\omega}{\sqrt{(1 + \omega^2)(100 + \omega^2)}}
$$

and

$$
\angle G(j\omega) = \frac{\pi}{2} - \left(\arctan(\omega) + \arctan \left(\frac{\omega}{10} \right) \right)
$$

Understanding these formulas allows engineers to analyze how the system responds to sinusoidal inputs across the frequency spectrum, facilitating effective control system design, stability analysis, and filtering applications. The insights gained from the magnitude and phase responses are integral in shaping the system behavior to meet desired specifications in various engineering fields.

Frequently Asked Questions

What is the general approach to find the magnitude and phase of the transfer function G(s) = s / [(s+1)(s+10)]?

The approach involves substituting s = j□ into G(s), then calculating the magnitude as |G(j□)| and the phase as angle G(j□). This involves computing the magnitude and phase of numerator and denominator separately and then combining them.

How do you compute the magnitude of G(s) = s / [(s+1)(s+10)] at a given frequency □?

Replace s with j□: G(j□) = j□ / [(j□+1)(j□+10)]. The magnitude is then |G(j□)| = |j□| / |(j□+1)(j□+10)| = $\square / (\square(\square^2+1)\ \square(\square^2+100))$.

How is the phase of G(s) = s / [(s+1)(s+10)] determined at a specific frequency □?

Calculate the phase as the sum of the phase of numerator minus the sum of the phases of the denominator. Specifically, phase G(j□) = 90° (from j□) - [phase of (j□+1) + phase of (j□+10)]. The phases of (j□+1) and (j□+10) are arctangent of □ over their respective real parts.

What is the phase contribution of the numerator s = j□ in G(s)?

The phase of j□ is 90°, since j□ lies purely on the imaginary axis.

How do you find the phase of each denominator term in G(s)?

For each term, such as (j□ + a), the phase is arctangent of (□ / a). For (j□+1), phase = arctangent(□ / 1) = arctangent(□). Similarly, for (j□+10), phase = arctangent(□ / 10).

At what frequency ω does $G(j\omega)$ have a magnitude of 1 (0 dB)?

Set $|G(j\omega)| = 1$: $\omega / (\sqrt{\omega^2+1}\ \sqrt{\omega^2+100}) = 1$. Solving this equation yields the cutoff frequency where the magnitude equals unity.

What is the phase of G(s) at low frequency (ω approaching 0)?

As ω approaches 0, the magnitude approaches 0, and the phase approaches - (arctangent(0) + arctangent(0)) = 0°, because the numerator's phase is 90°, but the denominator's phases approach 0°, resulting in a net phase of -90°.

What is the phase of G(s) at high frequency (ω approaching infinity)?

As ω approaches infinity, the phase approaches 90° (from numerator) minus (90° + 90°) from the denominator, resulting in a phase approaching -90°.

How can you graphically analyze the magnitude and phase of G(s) = s / [(s+1)(s+10)]?

Use Bode plots: plot $|G(j\omega)|$ in dB versus log ω for magnitude and phase versus log ω for phase. These plots help visualize the frequency response and identify cutoff frequencies and phase shifts.

Additional Resources

Analytical Expressions for the Magnitude and Phase of G(s) = \(\frac{s}{(s+1)(s+10)}\)

Understanding the behavior of transfer functions such as \(G(s) = \frac{s}{(s+1)(s+10)} \) is fundamental in control systems and signal processing. These functions characterize how systems respond to various inputs across different frequencies and are essential for stability analysis, system design, and performance optimization. In this comprehensive review, we delve into the derivation of the analytical expressions for both the magnitude and phase of \(G(s) \), providing detailed explanations to elucidate their significance and applications.

Introduction to Transfer Functions and Their Significance

Transfer functions serve as the mathematical backbone of linear time-invariant (LTI) systems, encapsulating the relationship between input and output signals in the Laplace domain. By analyzing $G(s)$, engineers and scientists can predict system behavior, design controllers, and ensure stability.

The specific transfer function under consideration,

$$
G(s) = \frac{s}{(s+1)(s+10)},
$$

exhibits a rational form with a zero at the origin and poles at $s = -1$ and $s = -10$. The zero influences the high-frequency gain, while the poles determine the system's stability and transient response. To understand how this system responds across frequencies, we analyze the magnitude and phase of $G(s)$ as functions of frequency.

Mathematical Foundations: Substituting $s = j\omega$

The first step in deriving the magnitude and phase expressions is to substitute the complex frequency variable s with $j\omega$, where ω is the angular frequency (rad/sec). This substitution aligns with the frequency response analysis, which examines system behavior along the imaginary axis in the complex plane.

$$
G(j\omega) = \frac{j\omega}{(j\omega + 1)(j\omega + 10)}.
$$

This substitution transforms the complex transfer function into a purely frequency-dependent form, enabling the calculation of magnitude and phase across the spectrum.

Analytical Expression for the Magnitude of $G(j\omega)$

The magnitude of $G(j\omega)$ reflects the system's gain at a specific frequency. It is given by the modulus of the complex function:

$$
|G(j\omega)| = \left| \frac{j\omega}{(j\omega + 1)(j\omega + 10)} \right|.
$$

Applying properties of complex numbers, this becomes:

$$
|G(j\omega)| = \frac{|j\omega|}{|j\omega + 1| \times |j\omega + 10|}.
$$

Let's analyze numerator and denominator separately:

Numerator:

$$
|j\omega| = \sqrt{(\text{Re}(j\omega))^2 + (\text{Im}(j\omega))^2} = \sqrt{0^2 + \omega^2} = |\omega|.
$$

Since frequency is non-negative, we consider:

\[
|j\omega| = \omega.
\]

Denominator:
For each factor:

\[
|j\omega + a| = \sqrt{(\text{Re}(j\omega + a))^2 + (\text{Im}(j\omega + a))^2}.
\]

Given \(j\omega + a = a + j\omega \):

- For \(j\omega + 1 \):

\[
|j\omega + 1| = \sqrt{1^2 + \omega^2} = \sqrt{1 + \omega^2}.
\]

- For \(j\omega + 10 \):

\[
|j\omega + 10| = \sqrt{10^2 + \omega^2} = \sqrt{100 + \omega^2}.
\]

Final Magnitude Expression:
Putting it all together,

\[

\boxed{

|G(j\omega)| = \frac{\omega}{\sqrt{1 + \omega^2} \times \sqrt{100 + \omega^2}}.

}

\]

This expression reveals how the magnitude varies with frequency, illustrating the system's gain characteristics across the spectrum.

Analytical Expression for the Phase of \(G(j\omega) \)

The phase of \(G(j\omega) \) indicates the phase shift introduced by the system at different frequencies, which is crucial in stability and feedback analysis. The phase of a complex function is the sum of the phases of numerator and denominator, considering their signs and quadrants.

The phase of \(G(j\omega) \), denoted \(\angle G(j\omega) \), can be expressed as:

\[

\angle G(j\omega) = \angle j\omega - \angle (j\omega + 1) - \angle (j\omega + 10).

\]

Step 1: Phase of the numerator \(j\omega \):

\[

\angle j\omega = \frac{\pi}{2} \quad \text{(or 90 degrees)} \quad \text{for } \omega > 0,

\]

and

\[

\angle j\omega = -\frac{\pi}{2} \quad \text{for } \omega < 0.

\]

In the context of frequency response analysis, \(\omega \geq 0 \), so:

\[
\boxed{
\angle j\omega = \frac{\pi}{2}.
}
\]

Step 2: Phase of the denominator terms:

- For \(j\omega + 1 \):

\[
\angle (j\omega + 1) = \arctan\left(\frac{\omega}{1}\right) = \arctan(\omega).
\]

- For \(j\omega + 10 \):

\[
\angle (j\omega + 10) = \arctan\left(\frac{\omega}{10}\right).
\]

Final phase expression:

\[
\boxed{
\angle G(j\omega) = \frac{\pi}{2} - \arctan(\omega) - \arctan\left(\frac{\omega}{10}\right).
}
\]

Expressed in degrees:

$$
\angle G(j\omega) = 90^\circ - \arctan(\omega) - \arctan\left(\frac{\omega}{10}\right),
$$

which provides a clear view of how phase shift varies with frequency.

Interpretation and Practical Implications

Having derived the analytical expressions for magnitude and phase, the next step involves interpreting these results within the context of system analysis.

Magnitude Behavior:

- At low frequencies ($\omega \to 0$), the magnitude:

$$
|G(j0)| = \frac{0}{\sqrt{1 + 0} \times \sqrt{100 + 0}} = 0,
$$

indicating negligible gain at DC. This suggests the system acts as a high-pass filter, emphasizing high-frequency signals.

- At very high frequencies ($\omega \to \infty$), the magnitude:

$$
|G(j\omega)| \sim \frac{\omega}{\omega \times \omega} = \frac{1}{\omega} \to 0,
$$

\]

implying attenuation at very high frequencies.

- The maximum gain occurs at intermediate frequencies, which can be located by analyzing the derivative of $|G(j\omega)|$ or through graphical methods such as Bode plots.

Phase Behavior:

- At low frequencies ($\omega \to 0$):

$$
\angle G(j0) = 90^\circ - 0^\circ - 0^\circ = 90^\circ,
$$

indicating a leading phase shift.

- At high frequencies ($\omega \to \infty$):

$$
\arctan(\omega) \to 90^\circ, \quad \arctan(\omega/10) \to 90^\circ,
$$

thus,

$$
\angle G(j\omega) \to 90^\circ - 90^\circ - 90^\circ = -90^\circ,
$$

indicating a phase lag at high frequencies.

This phase transition from +90° to -90° across the spectrum is characteristic of systems with zeros at the origin and poles on the real axis.

Applications and System Design Considerations

The analytical expressions for magnitude and phase are instrumental in several key areas:

1. Bode Plot Construction: Engineers often use these formulas to construct Bode plots manually or validate numerical simulations, providing insights into gain margins, phase margins, and stability.

2. Filter Design: Recognizing that \(G(s) \) exhibits high-pass characteristics guides its application in filtering high-frequency noise while attenuating low-frequency signals.

3. Stability Analysis: Phase and gain margins derived from these expressions inform the robustness of feedback control systems, aiding in the design of controllers that ensure stable operation.

4. Frequency Response Optimization: By understanding how the system behaves across frequencies, designers can tailor parameters to meet specific transient or steady-state requirements.

Conclusion

The derivation of analytical expressions for the magnitude and phase of \(G

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