A Nonhomogeneous Equation And A Particular Solution Are Given. Find A General Solution For The Equation.

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Understanding how to find the general solution of a differential equation when a nonhomogeneous equation and a particular solution are provided is a fundamental aspect of differential equations. This process involves combining the solutions of the associated homogeneous equation with the known particular solution of the nonhomogeneous equation. This article aims to explore this topic thoroughly, providing clarity on the steps involved, the underlying concepts, and practical methods to arrive at the complete general solution.

Introduction to Differential Equations

What Is a Differential Equation?

A differential equation is an equation that involves an unknown function and its derivatives. These equations are essential in modeling various phenomena in physics, engineering, biology, and other sciences. They describe how a quantity changes concerning another variable, often time or space.

Types of Differential Equations

Differential equations are broadly classified into:

- **Ordinary Differential Equations (ODEs):** Involving derivatives with respect to a single independent variable.
- **Partial Differential Equations (PDEs):** Involving derivatives with respect to multiple independent variables.

This discussion focuses on ordinary differential equations, specifically first and second-order linear equations.

Homogeneous vs. Nonhomogeneous Differential Equations

Homogeneous Differential Equations

A differential equation is homogeneous if it can be expressed in the form:
\[L(y) = 0 \]

where L is a linear operator acting on y. For example:
$$ y'' + p(x) y' + q(x) y = 0 $$
This type of equation has solutions called the complementary or homogeneous solutions.

Nonhomogeneous Differential Equations

A nonhomogeneous differential equation has a form:
$$ L(y) = g(x) $$
where $g(x)$ is a non-zero function. The presence of $g(x)$ makes the equation nonhomogeneous. An example:
$$ y'' + p(x) y' + q(x) y = f(x) $$
Solutions to such equations are composed of the homogeneous solution plus a particular solution.

Particular Solutions and Their Significance

What Is a Particular Solution?

A particular solution is a specific solution to a nonhomogeneous differential equation that satisfies the entire equation, including the non-zero right-hand side $g(x)$ or $f(x)$. It differs from the general solution of the homogeneous equation, which only satisfies the associated homogeneous part.

Methods to Find a Particular Solution

Common methods include:

1. Method of Undetermined Coefficients
2. Method of Variation of Parameters
3. Using known particular solutions for specific types of equations (e.g., exponential, polynomial, sinusoidal)

The method chosen depends on the form of $g(x)$.

Constructing the General Solution

The Fundamental Principle

The general solution to a nonhomogeneous differential equation is the sum of:

1. The general solution of the associated homogeneous equation, known as the complementary solution y_c
2. A particular solution y_p of the nonhomogeneous equation

Mathematically:
$$ y_{general} = y_c + y_p $$

Given Data and Objective

Suppose:
- The nonhomogeneous differential equation:
$$ y'' + p(x) y' + q(x) y = g(x) $$
- A particular solution:
$$ y_p $$
- The goal: Find the general solution y_{gen}.

Since the particular solution y_p is already provided, the process simplifies to determining the complementary solution y_c.

Step-by-Step Procedure to Find the General Solution

Step 1: Write Down the Homogeneous Equation

Extract the associated homogeneous equation:
$$ y'' + p(x) y' + q(x) y = 0 $$
This is obtained by setting the right-hand side to zero.

Step 2: Find the Complementary Solution y_c

Solve the homogeneous equation:
1. Assume a solution of the form:
$$ y = e^{rx} $$
2. Substitute into the homogeneous equation to obtain the characteristic equation:
$$ r^2 + p(x) r + q(x) = 0 $$

Note: For constant coefficients, $p(x)$ and $q(x)$ are constants, simplifying the process. For variable coefficients, methods like reduction of order or power series might be necessary.

3. Solve for roots r_1, r_2:
- Real and distinct roots lead to:
$$ y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x} $$
- Repeated roots:
$$ y_c = (C_1 + C_2 x) e^{r x} $$
- Complex roots:
$$ y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) $$

where $r = \alpha \pm i \beta$.

Step 3: Use the Given Particular Solution y_p

Since y_p is provided, it directly contributes to the general solution.

Step 4: Write the Complete General Solution

Combine the complementary solution y_c with y_p:

$$ y_{general} = y_c + y_p $$

This encompasses all possible solutions to the original nonhomogeneous differential equation.

Example Illustration

Given:

- Differential equation:

$$ y'' - 3 y' + 2 y = e^{x} $$

- Particular solution:

$$ y_p = \frac{1}{2} e^{x} $$

Step 1: Homogeneous Equation

$$ y'' - 3 y' + 2 y = 0 $$

Step 2: Find y_c

Characteristic equation:

$$ r^2 - 3 r + 2 = 0 $$

Roots:

$$ r = 1, 2 $$

So,

$$ y_c = C_1 e^{x} + C_2 e^{2x} $$

Step 3: Use the given particular solution y_p

$$ y_p = \frac{1}{2} e^{x} $$

Step 4: Write the general solution

$$
\boxed{
y_{general} = C_1 e^{x} + C_2 e^{2x} + \frac{1}{2} e^{x}
}
$$

Note that if the particular solution overlaps with the homogeneous solution (as in the case of e^{x}), the form of y_p would need adjustment (e.g., multiplying by x) to ensure linear independence. However, here, the given particular solution is already suitable.

Special Considerations and Tips

When the Particular Solution Coincides with Homogeneous Solutions

If the particular solution y_p is similar to the homogeneous solution (e.g., both involve e^{x}), the method involves multiplying the particular solution by x (or higher powers of x) until linear independence is achieved.

Using Variation of Parameters

When the method of undetermined coefficients fails or is complicated, variation of parameters offers a systematic way to find y_p. It involves:

- Using the solutions of the homogeneous equation to construct y_p.
- Integrating to find particular solutions tailored to the specific nonhomogeneous term.

Conclusion

Finding the general solution of a nonhomogeneous differential equation when a particular solution is known involves understanding the structure of the differential equation, solving the homogeneous part, and then combining it with the specific particular solution. The key insight is that the complete general solution is always the sum of the complementary (homogeneous) solution and any particular solution. This approach ensures that all solutions satisfying the differential equation are encompassed, providing a comprehensive understanding of the solution space.

By mastering these concepts and methods, one can efficiently solve a wide range of differential equations, aiding in modeling and problem-solving across numerous scientific disciplines.

Frequently Asked Questions

What is the general approach to finding the solution of a nonhomogeneous differential equation when a particular solution is already given?

The general solution is obtained by adding the complementary (homogeneous) solution to the given particular solution, i.e., $y(t) = y_h(t) + y_p(t)$. Since the particular solution is provided, only the homogeneous solution needs to be determined to complete the general solution.

How do you find the homogeneous solution when a particular solution is given for a nonhomogeneous

differential equation?

You solve the associated homogeneous differential equation (set the nonhomogeneous term to zero) to find the complementary solution y_h(t). This involves solving the differential equation without the nonhomogeneous part.

Why is it important to verify that the given particular solution is valid before constructing the general solution?

Because the particular solution must satisfy the original nonhomogeneous differential equation; verifying ensures that it is correct and that the general solution, which includes it, is valid.

Can the given particular solution be used directly to write the general solution? Why or why not?

Yes, if it satisfies the original nonhomogeneous equation, then the general solution can be written as the sum of this particular solution and the general solution of the homogeneous equation.

What types of differential equations commonly involve finding a particular solution, and how does knowing one simplify the process?

Linear nonhomogeneous differential equations often require finding a particular solution. Having a particular solution simplifies the process because it allows immediate construction of the general solution by adding the homogeneous solution.

How does the method of undetermined coefficients relate to given particular solutions in solving nonhomogeneous equations?

The method of undetermined coefficients is a systematic way to find a particular solution when the nonhomogeneous term is of a certain form. If a particular solution is already given, it indicates this step has been completed, and focus shifts to finding the homogeneous solution.

What is the significance of the constants in the homogeneous solution when constructing the general solution?

The constants in the homogeneous solution reflect the family of solutions to the homogeneous equation. When combined with the particular solution, they form the complete general solution, accounting for all possible solutions.

Additional Resources

A Nonhomogeneous Equation And A Particular Solution Are Given. Find A General Solution For The Equation.

When tackling differential equations, especially nonhomogeneous ones, understanding how to derive the general solution is fundamental. The problem typically involves two key components: a nonhomogeneous differential equation and a particular solution. Given these, the task is to find the complete or general solution that encompasses all possible solutions to the differential equation. This process combines the solutions to the homogeneous part with the particular solution, leading to a comprehensive understanding of the system's behavior.

Understanding Nonhomogeneous Differential Equations

Before delving into the solution process, it's essential to understand what a nonhomogeneous differential equation is. These are equations where the standard homogeneous form is altered by an additional, non-zero term often called the forcing term or nonhomogeneous part.

Homogeneous vs. Nonhomogeneous Equations

- Homogeneous Differential Equation:
An equation of the form
$$
L(y) = 0
$$
where L is a linear differential operator.

- Nonhomogeneous Differential Equation:
An equation of the form
$$
L(y) = g(x)
$$
where $g(x) \neq 0$. The function $g(x)$ introduces external forcing or input into the system.

Why is the distinction important?

The solution to a nonhomogeneous differential equation can be viewed as a combination of the complementary (or homogeneous) solution and a particular solution:
$$
y(x) = y_h(x) + y_p(x)
$$

Here:

- $y_h(x)$ solves the homogeneous equation $L(y) = 0$,
- $y_p(x)$ is a specific solution to the nonhomogeneous equation $L(y) = g(x)$.

The Strategy for Finding the General Solution

Given a nonhomogeneous differential equation and a particular solution, the process involves these key steps:

1. Identify the homogeneous equation and find its general solution.
2. Use the provided particular solution, which satisfies the nonhomogeneous

equation.
3. Combine the homogeneous solution with the particular solution to form the general solution.

In many cases, the particular solution is either provided or can be found using methods such as:

- Method of undetermined coefficients
- Variation of parameters
- Integrating factors (for first-order equations)

Once the particular solution y_p is known, the main task reduces to solving the homogeneous equation and then summing solutions.

Step-by-Step Guide to Deriving the General Solution

Step 1: Write Down the Differential Equation

Suppose you are given an explicit form such as:
$$
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)
$$

where:

- a_i are constants,
- $g(x)$ is the nonhomogeneous term.

Step 2: Find the Homogeneous Solution y_h

Solve the homogeneous counterpart:
$$
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0
$$

Method:

- Assume solutions of the form $y_h = e^{rx}$,
- Derive the characteristic equation:
$$
a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0
$$
- Find roots r_i.

Outcome:

- For real distinct roots r_i, the homogeneous solution is:
$$
y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots
$$
- For repeated roots or complex roots, modify accordingly.

Step 3: Incorporate the Particular Solution y_p

Given that a particular solution y_p is provided, you can directly include it in the total solution. Typically, y_p satisfies:
\[

$$
a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \dots + a_1 y_p' + a_0 y_p = g(x)
$$

Step 4: Write the General Solution

The general solution to the nonhomogeneous differential equation is:

$$
\boxed{
y(x) = y_h(x) + y_p(x)
}
$$

Where:

- $y_h(x)$ is the general solution to the homogeneous equation,
- $y_p(x)$ is the particular solution provided.

Example Illustration

Suppose you are given:

$$
y'' - 3 y' + 2 y = e^{x}
$$

and a particular solution:

$$
y_p = \frac{1}{2} e^{x}
$$

Step 1: Homogeneous Equation

$$
y'' - 3 y' + 2 y = 0
$$

Characteristic equation:

$$
r^2 - 3 r + 2 = 0
$$

$$
(r - 1)(r - 2) = 0
$$

$$
r = 1, 2
$$

Homogeneous solution:

$$
y_h = C_1 e^{x} + C_2 e^{2x}
$$

Step 2: Particular Solution

Given $y_p = \frac{1}{2} e^{x}$.

Step 3: General Solution

$$
\boxed{
y(x) = C_1 e^{x} + C_2 e^{2x} + \frac{1}{2} e^{x}
}
$$

Note that the particular solution y_p is part of the total solution, ensuring all solutions to the original nonhomogeneous equation are captured.

Additional Tips and Considerations

Handling Repeated Roots

- When roots of the characteristic equation are repeated, multiply the basic exponential solution by powers of x. For example, if r is a double root, the homogeneous solution includes e^{rx} and $x e^{rx}$.

When the Particular Solution is Not Provided

- Use methods like undetermined coefficients or variation of parameters to find y_p.

Validating the Solution

- Always verify that the sum $y_h + y_p$ satisfies the original differential equation.

Special Cases

- For equations with non-constant coefficients, solutions might require series solutions or numerical methods.
- For equations with oscillatory solutions, complex roots lead to sine and cosine terms.

Summary

In summary, when you are given a nonhomogeneous differential equation along with a particular solution, constructing the general solution involves:

- Solving the homogeneous part to find the general homogeneous solution y_h,
- Incorporating the provided particular solution y_p,
- Summing these to obtain the full solution:

$$
y(x) = y_h(x) + y_p(x)
$$

This approach guarantees that the general solution encompasses all possible behaviors of the system described by the differential equation. Mastering this process is essential for solving a wide range of physical, engineering, and mathematical problems where external forces or inputs influence system

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dynamics.

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Remember: The key is understanding the structure of the differential
equation, correctly solving the homogeneous part, and appropriately applying
the particular solution to complete the solution set. With practice, this
method becomes a powerful tool in your differential equations toolkit.
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