A Spherical Balloon Is Inflated So That Its Volume Is Increasing At The Rate Of 2.5 Ft3/min. How Rapidly

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Understanding how the size of a balloon changes as it is inflated is a classic problem in calculus, particularly in the field of related rates. When a spherical balloon is being inflated, its volume increases over time, and this change can be related to the rate at which its radius expands. This problem involves applying derivatives to relate the known rate of change of volume to the unknown rate of change of the radius.

In this article, we will explore the mathematical principles behind this problem, step-by-step, including how to derive the formulas involved, interpret the results, and apply the concepts to real-world scenarios. Whether you're a student preparing for calculus exams or a curious learner interested in related rates problems, this comprehensive guide will help you understand the process thoroughly.

Understanding the Problem Context

The problem states:

> A spherical balloon is being inflated so that its volume is increasing at a rate of 2.5 cubic feet per minute. How rapidly is the radius of the balloon increasing at this moment?

This type of question is typical in calculus, where the key is understanding how the change in one quantity (volume) relates to the change in another (radius). Since the balloon is spherical, its volume depends on its radius, and both quantities are changing over time.

Key points to note:

- The volume V of a sphere depends on its radius r as $V = \frac{4}{3} \pi r^3$.
- The rate of change of volume $\frac{dV}{dt}$ is given as 2.5 ft^3/min.
- The goal is to find the rate at which the radius is increasing $\frac{dr}{dt}$ at a specific moment when the volume is changing at this rate.

Mathematical Foundations: Formulas and Derivatives

To analyze this problem, we need to understand the relationships between the volume of a sphere and its radius, and how their rates of change relate.

Volume of a Sphere

The volume V of a sphere with radius r is given by the formula:

$$
V = \frac{4}{3} \pi r^3
$$

This formula expresses the volume explicitly in terms of the radius.

Differentiating Volume with Respect to Time

Since both V and r are functions of time t, we differentiate both sides of the volume formula with respect to t:

$$
\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}
$$

This is obtained by applying the chain rule:

$$
\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) = 4 \pi r^2 \frac{dr}{dt}
$$

This formula relates the rate of change of volume to the current radius and the rate of change of the radius.

Applying the Related Rates Concept

Related rates problems involve finding the rate of change of one quantity based on the rate of change of another. Here, we know:

- $\frac{dV}{dt} = 2.5$ ft^3/min
- $V = \frac{4}{3} \pi r^3$

And we need to find $\frac{dr}{dt}$.

Step-by-step approach:

1. Identify the knowns and unknowns:
- Known: $\frac{dV}{dt} = 2.5$ ft^3/min
- Unknown: $\frac{dr}{dt}$

2. Express the radius r in terms of the volume V:

\[

$$
r = \left(\frac{3V}{4 \pi} \right)^{1/3}
$$

However, since the rate of change is given in terms of V, it's more straightforward to use the derivative relation directly.

3. Use the related rates formula:

$$
\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}
$$

4. Solve for $\frac{dr}{dt}$:

$$
\frac{dr}{dt} = \frac{1}{4 \pi r^2} \frac{dV}{dt}
$$

Calculating the Rate of Radius Change

To proceed, we need the current radius r at the moment when $\frac{dV}{dt} = 2.5$ ft³/min. Since the problem doesn't specify the current volume or radius, it's typical to analyze the rate at a particular volume or radius.

Suppose we want to find the rate of radius change when the volume is a certain value V.

Given a specific volume V, the radius r can be calculated as:

$$
r = \left(\frac{3V}{4 \pi} \right)^{1/3}
$$

Once r is known, substitute into the formula for $\frac{dr}{dt}$.

Example: Calculating $\frac{dr}{dt}$ when the volume is 10 ft³

1. Calculate the radius:

$$
r = \left(\frac{3 \times 10}{4 \pi} \right)^{1/3} \approx \left(\frac{30}{4 \pi} \right)^{1/3}
$$

$$
r \approx \left(\frac{30}{12.566} \right)^{1/3} \approx (2.387)^{1/3} \approx 1.33\, \text{ft}
$$

2. Calculate $\frac{dr}{dt}$:

$$
\frac{dr}{dt} = \frac{1}{4 \pi r^2} \times 2.5
$$

$$
\frac{dr}{dt} = \frac{2.5}{4 \pi \times (1.33)^2} \approx \frac{2.5}{4 \pi \times 1.77} \approx \frac{2.5}{22.2} \approx 0.112\, \text{ft/min}
$$

This means that when the volume of the balloon is 10 ft^3, the radius is increasing at approximately 0.112 feet per minute.

General Formula for the Rate of Radius Change

In the most general case, without a specific volume or radius, the rate of change of the radius at any moment is:

$$
\boxed{
\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4 \pi r^2}
}
$$

Where:
- $\frac{dV}{dt}$ is the known rate at which volume increases,
- r is the current radius, which can be determined from the volume or measured directly.

Understanding the Impact of Different Volumes and Radii

The rate $\frac{dr}{dt}$ depends inversely on r^2. This implies:

- When the radius is small, the same rate of volume increase results in a larger $\frac{dr}{dt}$,
- When the radius is large, the radius increases more slowly even if the volume increases at the same rate.

This relationship illustrates a critical aspect of related rates: as a sphere gets larger, its radius grows more slowly for the same rate of volume increase.

Real-World Applications of Related Rates in Inflating Spheres

While this problem is theoretical, it has practical implications:

- Manufacturing and Design: Understanding how quickly a balloon or spherical container expands helps in designing inflation equipment.

- Medical Imaging: Similar principles apply when modeling the expansion of organs or tumors.
- Engineering: Predicting the behavior of gases in spherical tanks or balloons under various rates of inflation.

Additional Considerations and Extensions

1. Rate of Change of Surface Area

Since the surface area S of a sphere is $S = 4 \pi r^2$, its rate of change is:

$$
\frac{dS}{dt} = 8 \pi r \frac{dr}{dt}
$$

Knowing $\frac{dr}{dt}$, one can find how quickly the surface area increases during inflation.

2. Rate of Change of Other Quantities

- Surface Area change rate as shown above.
- Density or pressure changes if the material properties are considered.

3. Multiple Moments

Analyzing how the radius changes over different volume stages can help optimize inflation processes and material usage.

Summary and Key Takeaways

- The volume of a sphere relates to its radius as $V = \frac{4}{3} \pi r^3$.
- Differentiating with respect to time yields \(\frac{dV

Frequently Asked Questions

A spherical balloon's volume is increasing at a rate of 2.5 ft^3/min. How do you find the rate at which the radius of the balloon is increasing?

Use the volume formula for a sphere, $V = (4/3)\pi r^3$. Differentiate both sides with respect to time t to get $dV/dt = 4\pi r^2(dr/dt)$. Solve for dr/dt by dividing dV/dt by $4\pi r^2$: $dr/dt = (dV/dt) / (4\pi r^2)$. Plug in the known rate and the current radius to find dr/dt.

If the volume of the balloon increases at 2.5 ft^3/min, what is the rate of change of the radius when the radius is 3 ft?

First, compute $dr/dt = (dV/dt) / (4\pi r^2)$. Substitute $dV/dt = 2.5$ ft^3/min and r = 3 ft: $dr/dt = 2.5 / (4\pi(3)^2) = 2.5 / (4\pi\ 9) = 2.5 / (36\pi)$. Numerically, $dr/dt \approx 2.5 / 113.097 \approx 0.0221$ ft/min.

What is the significance of differentiating the volume formula in the context of inflating balloons?

Differentiating the volume formula allows us to relate the rates of change of volume and radius over time. It helps determine how quickly the radius of the balloon is expanding given the rate at which the volume is increasing.

How does the radius of a spherical balloon affect the rate at which its volume increases?

Since the volume of a sphere depends on the cube of its radius ($V \propto r^3$), larger radii imply that small increases in radius result in larger increases in volume. Therefore, as the radius increases, the rate of volume change for a given rate of radius increase also increases.

What are the key steps to solve a related rates problem involving a spherical balloon's volume and radius?

First, write the formula relating volume and radius, $V = (4/3)\pi r^3$. Differentiate both sides with respect to time to relate dV/dt and dr/dt. Substitute known values for dV/dt and r to find dr/dt. Ensure units are consistent and interpret the result in the context of the problem.

Additional Resources

Inflation Dynamics of a Spherical Balloon: An In-Depth Analysis

When it comes to understanding the fascinating world of geometric growth and rates of change, few topics capture the imagination quite like the inflation of a spherical balloon. Whether it’s a party favorite or a scientific demonstration, the process involves a complex interplay of volume, surface area, and radius — all evolving simultaneously. Today, we delve into the mathematical intricacies behind this process, specifically focusing on how quickly the radius of a spherical balloon increases when its volume is expanding at a given rate.

Understanding the Core Concept: Volume and Rate

of Change

At the heart of this exploration lies a fundamental question: How rapidly is the radius of a spherical balloon increasing when its volume is increasing at a known rate?

This question is rooted in the principles of calculus, particularly the concept of related rates. Related rates problems involve understanding how different quantities change over time as they are interconnected through a mathematical relationship.

The Geometry of a Sphere

A sphere’s volume V is given by the formula:

$$ V = \frac{4}{3} \pi r^3 $$

where:
- V is the volume,
- r is the radius of the sphere,
- π is a constant approximately equal to 3.14159.

As the balloon inflates, both V and r change with respect to time t. To analyze the rate at which r increases, we need to differentiate the volume formula with respect to time.

The Rate of Volume Change

Given data:
- $\frac{dV}{dt} = 2.5\, \text{ft}^3/\text{min}$

This signifies that every minute, the volume of the balloon increases by 2.5 cubic feet.

Deriving the Relationship Between Radius and Volume

To find how rapidly the radius increases, we employ differentiation techniques:

Differentiation of the Volume Formula

Starting with:

$$ V = \frac{4}{3} \pi r^3 $$

Differentiate both sides with respect to t:

$$ \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} $$

This step uses the chain rule, considering r as a function of t.

Isolating $\frac{dr}{dt}$:

Rearranged, the expression for the rate of change of the radius is:

$$ \frac{dr}{dt} = \frac{1}{4 \pi r^2} \frac{dV}{dt} $$

This formula reveals that the rate at which the radius increases depends on the current radius and the rate of volume increase.

Applying the Calculus: How Fast Is the Radius Increasing?

Now, to compute $\frac{dr}{dt}$, we need to know the current radius r of the balloon at the moment of interest. Since the problem states only the rate of volume increase, the typical approach is to analyze the rate at a specific volume or radius.

Scenario 1: Given a Specific Volume or Radius

Suppose we wish to find the rate when the volume is V or the radius is r. For the purpose of illustration, let's assume the current volume of the balloon is $V = 36\, \text{ft}^3$.

1. Calculate the current radius r:

Using the volume formula:

$$ r = \left(\frac{3V}{4 \pi} \right)^{1/3} $$

Plugging in $V = 36$:

$$ r = \left(\frac{3 \times 36}{4 \pi} \right)^{1/3} = \left(\frac{108}{4 \pi} \right)^{1/3} $$

$$ r = \left(\frac{27}{\pi} \right)^{1/3} $$

Numerically, with $\pi \approx 3.14159$:

$$ r \approx \left(\frac{27}{3.14159} \right)^{1/3} \approx \left(8.6 \right)^{1/3} $$

$$ r \approx 2.055\, \text{ft} $$

2. Calculate $\frac{dr}{dt}$:

Using the relation:

$$ \frac{dr}{dt} = \frac{1}{4 \pi r^2} \times 2.5 $$

Calculate r^2:

$$ r^2 \approx (2.055)^2 \approx 4.22 $$

Compute denominator:

\[4 \pi r^2 \approx 4 \times 3.14159 \times 4.22 \approx 4 \times 3.14159

\times 4.22 \approx 4 \times 13.27 \approx 53.09 \]

Finally,

\[\frac{dr}{dt} \approx \frac{2.5}{53.09} \approx 0.047\, \text{ft/min} \]

This indicates that, at the current volume, the radius of the balloon is increasing approximately 0.047 feet per minute.

Generalized Analysis: How the Rate Changes with Volume and Radius

The rate of change of radius is not constant; it depends on the current size of the balloon. As the balloon expands, the surface area increases, but at a non-linear rate, impacting \(\frac{dr}{dt} \).

Key Observations:

- Smaller radii: The rate at which the radius increases is higher when the balloon is smaller, because \(r^2 \) is smaller, making \(\frac{dr}{dt} \) larger.

- Larger radii: As the radius grows, \(r^2 \) increases, leading to a decrease in \(\frac{dr}{dt} \). This means the balloon's radius grows more slowly as it gets larger, even if the volume continues to increase at the same rate.

Practical Implications:

- When inflating a balloon at a constant volumetric rate, the rate of radius increase diminishes over time.
- For manufacturers or hobbyists, understanding this behavior is crucial for controlling inflation speed, especially in delicate applications.

Real-World Applications and Significance

While this analysis seems purely mathematical, it has tangible real-world applications:

1. Designing Inflatable Structures

Engineers designing inflatable habitats, emergency shelters, or scientific balloons need to precisely control the inflation rate to ensure structural integrity and safety. Knowing how the radius changes allows for better control over inflation processes.

2. Educational Demonstrations

Mathematics educators can leverage such related rates problems to demonstrate calculus concepts, making abstract ideas concrete through real-world examples.

3. Event Planning and Decorations

Event organizers and decorators often inflate balloons at controlled rates to prevent popping or deformation, especially for large installations.

4. Scientific Research

Scientists studying atmospheric or space balloons require accurate models of inflation dynamics to predict behavior during ascent and expansion.

Summary and Final Thoughts

In essence, the process of a spherical balloon inflating at a rate of 2.5 ft^3/min reveals fascinating insights into the interconnectedness of volume, radius, and their rates of change. By understanding the derivatives involved, we can predict how quickly the radius expands at any given moment, which is crucial for both practical applications and theoretical comprehension.

Key Takeaways:

- The volume V of a sphere relates to its radius r via $V = \frac{4}{3} \pi r^3$.
- Differentiating with respect to time yields a relationship: $\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}$.
- To find $\frac{dr}{dt}$, rearrange as $\frac{dr}{dt} = \frac{1}{4 \pi r^2} \frac{dV}{dt}$.
- The radius's rate of increase diminishes as the balloon gets larger because it’s inversely proportional to r^2.

Whether you’re a mathematician, engineer, or enthusiast, understanding these relationships enhances your grasp of how geometric objects grow and change, emphasizing the elegance and utility of calculus in solving real-world problems.

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