(b) Consider The Steady, Two-dimensional Velocity Field Given By (i) (ii) V = (x - Y) I - 2xyj Determine

(b) Consider The Steady, Two-dimensional Velocity Field Given By (i) (ii) V = (x - Y) I - 2xyj Determine

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Introduction

Understanding velocity fields in fluid dynamics is fundamental to analyzing how fluids move under various conditions. In this article, we examine a specific steady, two-dimensional velocity field defined by the vector function $\mathbf{V} = (x - y)\mathbf{i} - 2xy \mathbf{j}$. Our goal is to determine various properties associated with this velocity field, including divergence, curl, flow patterns, streamlines, and potential functions if applicable. These analyses shed light on the physical implications of the flow, such as whether it is incompressible, irrotational, or exhibits specific flow features.

Mathematical Preliminaries and Notation

Before delving into the analysis, it is essential to clarify the notation and the mathematical tools used:

- $\mathbf{V} = u(x, y) \mathbf{i} + v(x, y) \mathbf{j}$, where $u(x, y)$ and $v(x, y)$ are the velocity components in the x and y directions, respectively.
- The divergence of $\mathbf{V}$: $\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$.
- The curl (specifically the scalar curl in 2D): $\nabla \times \mathbf{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.
- Streamlines are curves tangent to the velocity field at every point, satisfying $\frac{dy}{dx} = \frac{v}{u}$.
- Potential functions are scalar functions $\phi(x, y)$ such that $\mathbf{V} = \nabla \phi$ if the flow is irrotational.

Expressing the Velocity Field Components

Given the velocity field:

$$
\mathbf{V} = (x - y)\mathbf{i} - 2xy \mathbf{j}
$$

we identify:

$$
u(x, y) = x - y
$$
$$
v(x, y) = -2xy
$$

These components form the basis for subsequent calculations.

Calculating the Divergence of the Velocity Field

The divergence measures the net rate of fluid expansion or compression at a point. For the given field:

$$
\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
$$

Calculations:

$$
\frac{\partial u}{\partial x} = \frac{\partial (x - y)}{\partial x} = 1
$$
$$
\frac{\partial v}{\partial y} = \frac{\partial (-2xy)}{\partial y} = -2x
$$

Thus,

$$
\nabla \cdot \mathbf{V} = 1 - 2x
$$

Physical interpretation:

- The divergence depends on x. When $x = 0.5$, divergence is zero, indicating locally incompressible flow.
- For $x < 0.5$, divergence is positive, indicating local expansion.
- For $x > 0.5$, divergence is negative, indicating compression.

Implication:

Since the divergence is not zero everywhere, the flow is generally compressible, with regions of expansion and compression depending on x.

Calculating the Curl of the Velocity Field

In two dimensions, the scalar curl is given by:

$$
\nabla \times \mathbf{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
$$

Calculations:

$$
\frac{\partial v}{\partial x} = \frac{\partial (-2xy)}{\partial x} = -2y
$$

$$
\frac{\partial u}{\partial y} = \frac{\partial (x - y)}{\partial y} = -1
$$

Therefore,

$$
\nabla \times \mathbf{V} = -2y - (-1) = -2y + 1
$$

Physical interpretation:

- The vorticity (rotation per unit area) varies linearly with y.
- At $y = 0.5$, the curl is 0, indicating no local rotation.
- For $y > 0.5$, the vorticity is negative, indicating clockwise rotation.
- For $y < 0.5$, the vorticity is positive, indicating counterclockwise rotation.

Flow Pattern and Streamlines

Understanding the flow lines or streamlines involves solving the differential equation:

$$
\frac{dy}{dx} = \frac{v}{u} = \frac{-2xy}{x - y}
$$

This differential equation describes the path of fluid particles.

Deriving the Streamline Equation

Rearranged differential equation:

$$
\frac{dy}{dx} = \frac{-2xy}{x - y}
$$

To solve, perform substitution:

Let $y = kx$, where k is a constant, to examine particular solutions.

Substituting:

$$
y = kx \Rightarrow \frac{dy}{dx} = k
$$

Plug into the differential equation:

$$
k = \frac{-2x(kx)}{x - kx} = \frac{-2k x^2}{x(1 - k)} = \frac{-2k x}{1 - k}
$$

Rearranged:

$$
k(1 - k) = -2k x
$$

Dividing both sides by k (assuming $k \neq 0$):

$$
1 - k = -2x
$$

Thus,

$$
x = \frac{k - 1}{2}
$$

This indicates that for a particular slope k, the streamline is a straight line $y = kx$ at specific x.

Alternatively, for a more general solution, rewrite the differential equation as:

$$
\frac{dy}{dx} = \frac{-2xy}{x - y}
$$

Let $z = y/x$, then:

$$
y = z x
$$

$$
\frac{dy}{dx} = z + x \frac{dz}{dx}
$$

\]

Substitute into the differential equation:

$$
z + x \frac{dz}{dx} = \frac{-2x(zx)}{x - z x} = \frac{-2z x^2}{x (1 - z)} = \frac{-2z x}{1 - z}
$$

Rearranged:

$$
z + x \frac{dz}{dx} = \frac{-2z x}{1 - z}
$$

Subtract z:

$$
x \frac{dz}{dx} = \frac{-2z x}{1 - z} - z
$$

Express the right side over common denominator:

$$
x \frac{dz}{dx} = \frac{-2z x - z (1 - z)}{1 - z}
$$

Simplify numerator:

$$
-2z x - z + z^2
$$

Thus:

$$
x \frac{dz}{dx} = \frac{-2z x - z + z^2}{1 - z}
$$

This differential equation can be solved further to find explicit streamlines, but the key insight is that the flow exhibits a complex pattern with streamlines approximated by these solutions.

Potential Function and Irrotationality

A flow is irrotational if the curl is zero throughout the flow field, i.e.,

$$
\nabla \times \mathbf{V} = 0
$$

From earlier:

$$
\nabla \times \mathbf{V} = -2 y + 1

\]

Set:

\[
-2 y + 1 = 0 \Rightarrow y = \frac{1}{2}
\]

This indicates the flow is irrotational only along the line \(y = 0.5\). Elsewhere, the flow is rotational.

Existence of a Velocity Potential \(\phi(x, y)\)

Since the flow is not irrotational everywhere, a global potential function \(\phi\) does not exist for the entire field. However, along the line \(y = 0.5\), the flow is irrotational, and potential functions may be constructed locally.

For regions where \(\nabla \times \mathbf{V} \neq 0\):

\[
\frac{\partial \phi}{\partial x} = u = x - y
\]
\[
\frac{\partial \phi}{\partial y} = v = -2xy
\]

Attempting to find \(\phi\):

\[
\phi(x, y) = \int u \, dx

Frequently Asked Questions

What is the given steady, two-dimensional velocity field in the problem?

The velocity field is given by V = (x - y) i - 2xy j.

How can we verify if the velocity field is divergence-free?

Calculate the divergence: ∇·V = ∂/∂x (x - y) + ∂/∂y (-2xy). This simplifies to 1 + (-2x), which is zero only when x = 0. Generally, the field is not divergence-free.

What does the form of the velocity field suggest about the flow pattern?

The field combines linear and quadratic terms, indicating complex flow patterns such as saddle points or vortices depending on the location in the domain.

How do we compute the stream function for this velocity field?

For incompressible flow, the stream function ψ satisfies $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Using these, integrate to find ψ: from $u = x - y$, $\psi = x y + C$; from $v = -2xy$, integrating accordingly, to identify the stream function.

Is the velocity field incompressible, and how can we confirm?

Calculate divergence: $\nabla\cdot V = 1 - 2x$. Since this isn't zero everywhere, the flow is generally compressible except at $x = 0$.

How can we determine the acceleration field for this velocity?

Use the material derivative: $a = (V \cdot \nabla) V$. Compute the derivatives of V and evaluate to find the acceleration components at any point (x, y).

What is the significance of analyzing the flow at specific points, such as (0,0)?

At (0,0), the velocity $V = (0,0)$, indicating a stagnation point, which helps understand flow separation or stagnation zones.

How can this velocity field be used to study flow stability or streamline patterns?

By plotting the stream function or streamlines derived from the velocity field, we can analyze flow patterns and assess stability or presence of vortices.

What mathematical methods are most useful for analyzing this velocity field?

Vector calculus techniques such as divergence, curl, potential functions, stream functions, and flow visualization methods are essential for analyzing the field.

Could this velocity field represent a physical flow, and what applications might it have?

Yes, it can model certain idealized flows in fluid dynamics, such as saddle flows or flow around objects, relevant in engineering and physics simulations.

Additional Resources

Consider The Steady, Two-dimensional Velocity Field Given By (i) (ii) V = (x – Y) I – 2xyj Determine

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Introduction

In fluid mechanics and vector calculus, understanding the behavior of velocity fields is fundamental to analyzing the motion of fluids and other continuum systems. When such a velocity field is steady and two-dimensional, it possesses specific characteristics that simplify analysis and reveal underlying flow structures. The velocity field under consideration, given by V = (x - y)i - 2xyj, presents an interesting case for exploration. This article offers a comprehensive analysis of this velocity field, examining its properties, flow patterns, mathematical features, and physical implications.

---

1. Overview of the Velocity Field

1.1. Mathematical Expression

The velocity field under scrutiny is expressed as:

$$
\mathbf{V}(x, y) = (x - y) \mathbf{i} - 2xy \mathbf{j}
$$

where:

- x, y are the Cartesian coordinates in the plane,
- $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions, respectively.

This representation indicates that the flow velocity at any point (x, y) is determined by the position within the plane, with the x-component dependent on the difference $x - y$, and the y-component dependent on the product $-2xy$.

1.2. Assumptions and Conditions

The problem specifies a steady velocity field, implying that:

- The velocity does not vary with time.
- The flow is time-independent, leading to potential simplifications in the analysis.

Additionally, the field is two-dimensional, confined to the xy-plane, which simplifies the mathematical treatment and physical interpretation.

---

2. Physical Significance of the Velocity Field

2.1. Flow Characteristics

Understanding the physical nature of this velocity field involves examining the flow pattern, stagnation points, and potential sources or sinks.

- Flow Pattern: The flow pattern can be visualized by plotting the velocity vectors at various points in the plane.
- Stagnation Points: Locations where the velocity magnitude becomes zero are critical points, often indicating flow separation or stagnation zones.
- Sources and Sinks: The divergence of the velocity field indicates whether the flow behaves as a source, sink, or incompressible flow.

2.2. Applications and Relevance

While hypothetical, analyzing such a field is valuable in theoretical fluid mechanics, especially in understanding flow behavior in idealized conditions, such as potential flows, or in the study of flow topology.

---

3. Mathematical Analysis of the Velocity Field

3.1. Divergence and Its Physical Interpretation

The divergence of the velocity field measures the net outflow or inflow at a point:

$$
\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
$$

where $u = x - y$ and $v = -2xy$.

Calculating:

$$
\frac{\partial u}{\partial x} = \frac{\partial (x - y)}{\partial x} = 1
$$
$$
\frac{\partial v}{\partial y} = \frac{\partial (-2xy)}{\partial y} = -2x
$$

Thus,

$$
\nabla \cdot \mathbf{V} = 1 - 2x
$$

Implication: The divergence depends on the x-coordinate, indicating that the flow is compressible or incompressible depending on the region in the plane.

3.2. Vorticity and Flow Rotation

Vorticity describes the local rotation of fluid elements:

$$
\boldsymbol{\omega} = \nabla \times \mathbf{V}
$$

In 2D, the vorticity reduces to a scalar:

$$
\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
$$

Calculating:

$$
\frac{\partial v}{\partial x} = \frac{\partial (-2xy)}{\partial x} = -2y
$$

$$
\frac{\partial u}{\partial y} = \frac{\partial (x - y)}{\partial y} = -1
$$

Therefore,

$$
\omega_z = -2y - (-1) = -2y + 1
$$

Implication: The vorticity varies linearly with y, indicating regions of rotational flow with the sense and magnitude changing vertically.

3.3. Stream Function and Potential Function

The stream function $\psi(x, y)$ and potential function $\phi(x, y)$ are useful tools for visualizing flow:

$$
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
$$

and

$$
u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}
$$

Given $u = x - y$ and $v = -2xy$:

- For the stream function:

$$
\frac{\partial \psi}{\partial y} = x - y
$$

Integrating with respect to y:

$$
\psi(x, y) = xy - \frac{1}{2} y^2 + C(x)
$$

- For the potential function:

$$
\frac{\partial \phi}{\partial x} = x - y
$$

$$
\Rightarrow \phi(x, y) = \frac{1}{2} x^2 - xy + D(y)

\]

Similarly, verifying the consistency and physical meaning of these functions aids in understanding the flow's qualitative behavior.

---

4. Flow Topology and Critical Points

4.1. Identification of Critical Points

Critical points are where the velocity components vanish:

$$
u = x - y = 0 \quad \Rightarrow y = x
$$
$$
v = -2xy = 0
$$

The second condition implies either:

- $x = 0$, or
- $y = 0$

From the first condition $y = x$, substituting into the second:

- When $x = 0$, then $y=0$: the origin.

- When $y=0$, then $x=0$: again, the origin.

Conclusion: The only critical point is at $(0, 0)$.

4.2. Nature of the Critical Point

To analyze the flow near $(0, 0)$, linearization can be employed by examining the Jacobian matrix:

$$
J = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
1 & -1 \\
-2y & -2x
\end{bmatrix}
$$

At $(0, 0)$, this reduces to:

$$
J(0, 0) = \begin{bmatrix}
1 & -1 \\
0 & 0
\end{bmatrix}
$$

Eigenvalues are obtained from:

$$
\det(J - \lambda I) = 0
$$

which simplifies to:

$$
(1 - \lambda)(-\lambda) = 0
$$

yielding eigenvalues:

$$
\lambda_1 = 1, \quad \lambda_2 = 0
$$

This indicates a saddle or degenerate critical point, with flow behavior characterized accordingly.

---

5. Physical Interpretation and Visualization

5.1. Flow Patterns and Streamlines

The streamlines, or flow trajectories, are obtained by solving:

$$
\frac{dy}{dx} = \frac{v}{u} = \frac{-2xy}{x - y}
$$

This differential equation describes the shape of flow paths. Analytic solutions may be complex, but qualitative analysis and numerical visualization reveal key features:

- Flow separation points: The critical point at the origin acts as a stagnation point.
- Directionality: The flow exhibits regions of convergence and divergence based on the divergence calculation.
- Rotational behavior: The vorticity indicates swirling or rotational flow in certain regions.

5.2. Visualizing the Flow

Plotting vector fields and streamlines can elucidate the flow topology. Such visualization reveals:

- Flow toward or away from critical points.
- Vortical regions where rotation dominates.
- Flow bifurcations or saddle points indicating complex topologies.

5.3. Physical Implications

While the defined velocity field is idealized, it models fundamental flow features such as:

- Stagnation points where fluid velocity is zero.
- Vorticity zones indicating rotational behavior.
- Flow separation regions that could relate to real-world phenomena like

boundary layer detachment.

---

6. Conservation Laws and Flow Properties

6.1. Incompressibility

The divergence of the flow determines whether the fluid is incompressible:

```
\[
\nabla \cdot \mathbf{V} = 1 -
```

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