Compute The Kernel For Each Of The Following Homomorphisms . (a) :ZZ Such That (1)=12 (b) :ZZZ Such That

Compute The Kernel For Each Of The Following Homomorphisms . (a) :ZZ Such That (1)=12 (b) :ZZZ Such That

Understanding homomorphisms and their kernels is fundamental in abstract algebra, particularly in the study of group theory. A homomorphism is a structure-preserving map between two algebraic structures, such as groups, rings, or modules. The kernel of a homomorphism captures the elements that map to the identity element in the target structure, revealing key properties about the homomorphism itself and the structure of the domain. This article provides a comprehensive guide to computing the kernels for specific homomorphisms involving the integers (denoted as ZZ) and their variants, illustrating how to analyze these mappings systematically.

Overview of Homomorphisms and Kernels

Before delving into specific examples, it's essential to review foundational concepts.

What Is a Homomorphism?

A homomorphism is a function $\varphi : G \to H$ between two groups G and H such that for all elements $a, b \in G$,

$$
\varphi(ab) = \varphi(a) \varphi(b).
$$

\]

This property ensures the algebraic structure (like operation rules) is preserved under the mapping.

What Is the Kernel of a Homomorphism?

The kernel of a homomorphism $\varphi : G \to H$ is the set of all elements in G that map to the identity element in H:

$$
\ker(\varphi) = \{ g \in G \mid \varphi(g) = e_H \}.
$$

The kernel is always a normal subgroup of G, and it measures the "loss of information" caused by the homomorphism.

Relevance of Computing Kernels

Calculating the kernel can tell us:

- Whether the homomorphism is injective (one-to-one).
- The structure of the quotient group $G / \ker(\varphi)$.
- How the domain group maps onto the image.

Now, let’s analyze the specific cases provided.

Case (a): Homomorphism \(\varphi: \mathbb{Z} \to

$\mathbb{Z}$ Such That $\varphi(1) = 12$

This is a classic example of a group homomorphism from the integers to themselves, defined by the image of the generator 1. Since $\mathbb{Z}$ is generated by 1, the entire homomorphism is determined by $\varphi(1)$.

Constructing the Homomorphism

Given $\varphi(1) = 12$, the homomorphism $\varphi: \mathbb{Z} \to \mathbb{Z}$ can be expressed as:

$$
\varphi(n) = n \cdot \varphi(1) = 12n,
$$

for all integers n. This is because any integer n can be written as $n \times 1$, and the homomorphism respects addition:

$$
\varphi(n) = \varphi(1 + 1 + \dots + 1) \text{ (n times)} = \underbrace{\varphi(1) + \varphi(1) + \dots + \varphi(1)}_{n \text{ times}} = 12n.
$$

Note: For negative n, the same logic applies, with $\varphi(n) = 12n$.

Calculating the Kernel

The kernel consists of all integers n such that $\varphi(n)$ is the identity element in the codomain $\mathbb{Z}$, which is 0 under addition:

$$
\ker(\varphi) = \{ n \in \mathbb{Z} \mid \varphi(n) = 0 \}.
$$

Using $\varphi(n) = 12n$, the kernel becomes:

$$
\ker(\varphi) = \{ n \in \mathbb{Z} \mid 12n = 0 \}.
$$

This implies:

$$
12n = 0 \Rightarrow n = 0,
$$

since 12 is non-zero and the integers form an integral domain.

Result:

$$
\boxed{
\ker(\varphi) = \{ 0 \}
}
$$

This shows that the kernel is trivial, meaning the homomorphism is injective.

Summary for Case (a)

- The homomorphism $\varphi: \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(1) = 12$ is given by

$\varphi(n) = 12n$.

- Its kernel is $\{ 0 \}$, indicating the map is injective.
- The image of φ is the subgroup $12\mathbb{Z}$ of $\mathbb{Z}$.

Case (b): Homomorphism $\varphi: \mathbb{Z}_Z \to \text{(target group)}$ Such That

The description for case (b) appears incomplete in the prompt, but a typical scenario involves homomorphisms from a cyclic group $\mathbb{Z}_n$, the integers modulo n, into another group.

Assuming the intended homomorphism is from $\mathbb{Z}_n$ to some group G, the structure of the kernel depends on the specific map and the properties of the target group.

Note: Since the original prompt is incomplete ("Such That" without specifics), we will analyze a common case:

Hypothetical Scenario: Homomorphism $\varphi: \mathbb{Z}_n \to G$

Suppose we have a homomorphism φ from the cyclic group $\mathbb{Z}_n$ (integers mod n) into some group G. The key facts about such homomorphisms include:

- The homomorphism is determined by the image of the generator 1 in $\mathbb{Z}_n$, which has order n.
- The kernel is a subgroup of $\mathbb{Z}_n$, hence cyclic, and is determined by the order of the image of the generator.

Computing the Kernel in the General Case

Given:

$$
\varphi: \mathbb{Z}_n \to G,
$$

and $\varphi(1) = g \in G$, then:

$$
\varphi(k) = g^k,
$$

where the operation in G is written multiplicatively, or simply $\varphi(k)$ is determined accordingly.

The kernel consists of all elements $k \in \mathbb{Z}_n$ such that:

$$
\varphi(k) = e_G,
$$

the identity in G.

Because $\mathbb{Z}_n$ is cyclic, the subgroup structure is straightforward:

$$
\ker(\varphi) = \{ k \in \mathbb{Z}_n \mid g^k = e_G \}.
$$

The order of g, say d, divides n, and the kernel consists of all multiples of d:

$$
\ker(\varphi) = \langle d \rangle,
$$

which is a subgroup isomorphic to $\mathbb{Z}_{n/d}$.

Specific Calculation of the Kernel

Suppose the image of the generator has order m, then:

$$
\ker(\varphi) = \{ k \in \mathbb{Z}_n \mid m \mid k \}.
$$

The kernel is a subgroup of $\mathbb{Z}_n$, specifically:

$$
\ker(\varphi) \cong \mathbb{Z}_{\gcd(n, m)}.
$$

In particular:

- If g is the identity e_G, then the kernel is the whole group $\mathbb{Z}_n$.
- If g has maximal order n, then the kernel is trivial.

Summary for Case (b)

- To compute the kernel, determine the order of the image of the generator.
- The kernel is a subgroup of \(\mathbb{Z}_n \), typically cyclic.
- The order of the kernel divides \(n \) and is related to the order of the image element.

Summary and Final Remarks

In this article, we've examined how to compute the kernels of specific homomorphisms involving integers and cyclic groups.

For the homomorphism \(\varphi: \mathbb{Z} \to \mathbb{Z} \) with \(\varphi(1) = 12 \):

- The homomorphism is given by \(\varphi(n) = 12n

Frequently Asked Questions

What is the kernel of the homomorphism □: □ □ □ defined by □(1) = 12?

The kernel of □ is the set of all integers n such that □(n) = 0, which is n □ 12□. Therefore, ker(□) = 12□.

How do you compute the kernel of a homomorphism from □ to □ given

by □(1) = 12?

Since □ is determined by □(1) = 12, for any n □ □, □(n) = 12n. The kernel consists of all n where 12n = 0, which is only n = 0, but in the context of homomorphisms from □, the kernel is all n such that 12 divides n, i.e., n □ 12□.

What is the kernel of a homomorphism □: □ □ □□z□ such that □(1) = a?

The kernel consists of all integers n where □(n) = 0 in □□. Since □(n) = n · a mod n, the kernel is all n such that n · a □ 0 mod n, which simplifies to n dividing a · n, so the kernel depends on the specific value of a and the structure of the homomorphism.

In the case of the homomorphism □: □ □ □ with □(1) = 12, why is the kernel equal to 12□?

Because □(n) = 12n, and □(n) = 0 when 12n = 0, which in □ occurs only when n is a multiple of 12. Hence, the kernel is all n such that n □ 12□.

If a homomorphism from □ to an abelian group G is defined by sending 1 to an element g □ G, how is the kernel related to the order of g?

The kernel consists of all integers n such that n · g = 0 in G. If g has finite order m, then the kernel is n · g = 0 iff m divides n, so the kernel is m□.

For the homomorphism □: □ □ □ defined by □(n) = 12n, what does the kernel tell us about the divisibility properties of integers?

The kernel being 12□ indicates that all integers divisible by 12 map to zero, reflecting the divisibility condition that characterizes the kernel.

Can the kernel of a homomorphism from □ to □ be any subgroup of □?

No, since □ is cyclic, all subgroups are of the form n□ for some n □ 0. The kernel of a homomorphism from □ to □ is always of the form n□, where n is determined by the image of 1.

Additional Resources

Compute the Kernel for Each of the Following Homomorphisms is a fundamental task in abstract algebra, particularly within the study of group theory and module theory. Understanding the kernel of a homomorphism provides insight into the structure of the groups involved, revealing elements that map to the identity and thus shedding light on the ways these groups relate to each other. In this guide, we will explore how to compute the kernels of specific homomorphisms involving the groups $\mathbb{Z}$ and $\mathbb{Z}_3$. The examples include homomorphisms from $\mathbb{Z}$ to other groups, with particular emphasis on the case where the homomorphism is characterized by specific images of the generator 1.

Introduction to Kernels of Homomorphisms

In algebra, a homomorphism is a structure-preserving map between two algebraic structures such as groups, rings, or modules. When dealing with groups, a homomorphism $\varphi: G \to H$ respects the group operation:

$$
\varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2) \quad \text{for all } g_1, g_2 \in G.
$$

The kernel of a homomorphism, denoted $\ker \varphi$, is the set of elements in G that map to the identity element in H:

$$
\ker \varphi = \{ g \in G \mid \varphi(g) = e_H \}.
$$

The kernel is always a normal subgroup of G, and understanding it is crucial because it measures how far the homomorphism is from being injective. If the kernel is trivial (contains only the identity element), then the homomorphism is injective.

Computing the Kernel of Homomorphisms from $\mathbb{Z}$

The Group $\mathbb{Z}$ and Its Homomorphisms

The group $\mathbb{Z}$, under addition, is a fundamental object in algebra. Any homomorphism from $\mathbb{Z}$ to an arbitrary group G is determined entirely by the image of 1, since:

$$
\varphi(n) = \varphi(1 + 1 + \cdots + 1) = n \varphi(1).
$$

Thus, to understand a homomorphism $\varphi : \mathbb{Z} \to G$, it suffices to analyze $\varphi(1)$.

Example (a): Homomorphism $\varphi: \mathbb{Z} \to G$ such that $\varphi(1) = 12$

Step 1: Define the Homomorphism

Given the condition $\varphi(1) = 12$, the homomorphism φ is completely specified as:

$$
\varphi(n) = n \times 12.
$$

This map takes an integer n and maps it to the element $12n$ in G.

Step 2: Determine the Kernel

The kernel consists of all integers n such that:

$$
\varphi(n) = 0_G,
$$

where 0_G is the identity element in the target group G. Assuming G is $\mathbb{Z}$ itself or an abelian group where the operation is addition, then:

$$
\varphi(n) = 12n = 0.
$$

The solutions to this are all integers n such that:

$$
12n = 0,
$$

which implies:

$$
n = 0,
$$

if $G = \mathbb{Z}$.

However, if G is a different group, the structure of the kernel depends on the nature of G and whether the element 12 is mapped into an element of finite order or not.

Step 3: Kernel in $\mathbb{Z}$

In the most straightforward case where $G = \mathbb{Z}$, the kernel is:

$$
\ker \varphi = \{ n \in \mathbb{Z} \mid 12n = 0 \} = \{ 0 \}.
$$

This indicates that the homomorphism is injective in this case because the kernel is trivial.

Summary for Example (a):

- The kernel of the homomorphism $\varphi: \mathbb{Z} \to \mathbb{Z}$ with $\varphi(1) = 12$ is trivial:

$$
\boxed{
\ker \varphi = \{ 0 \}
}
$$

- Note: If the target group G is different, the kernel's structure could change accordingly, especially if G has elements of finite order.

Example (b): Homomorphism $\varphi: \mathbb{Z}_3 \to G$ such that...

(Note: The original question appears truncated, but commonly, such examples involve homomorphisms from $\mathbb{Z}_n$ groups to other groups.)

Step 1: Understand $\mathbb{Z}_3$

The group $\mathbb{Z}_3 = \{ 0, 1, 2 \}$ under addition modulo 3 is cyclic of order 3. Any homomorphism from $\mathbb{Z}_3$ to another group G is determined by the image of 1, due to the cyclic nature:

$$
\varphi(1) \in G,
$$

and

$$
\varphi(2) = 2 \varphi(1).
$$

The structure of the kernel depends on the element $\varphi(1)$.

Step 2: General Form of Homomorphisms from $\mathbb{Z}_3$

Suppose we define $\varphi: \mathbb{Z}_3 \to G$ such that:

\[

$$
\varphi(1) = g \in G,
$$

then

$$
\varphi(n) = n g,
$$

for all $n \in \mathbb{Z}_3$, where the addition is modulo 3.

Step 3: Computing the Kernel

The kernel is:

$$
\ker \varphi = \{ n \in \mathbb{Z}_3 \mid \varphi(n) = e_G \}.
$$

Since $\mathbb{Z}_3$ is finite, we check each element:

- $n = 0$:

$$
\varphi(0) = 0 \times g = e_G,
$$

so $0 \in \ker \varphi$.

- $n = 1$:

$$
\varphi(1) = g,
$$

which equals e_G if and only if $g = e_G$.

- $n = 2$:

$$
\varphi(2) = 2g,
$$

which equals e_G if and only if $2g = e_G$.

Step 4: Conditions for the Kernel

- If $g = e_G$, then

$$
\ker \varphi = \mathbb{Z}_3,
$$

since all elements map to the identity.

- If $g \neq e_G$, then the kernel consists only of the element 0:

$$
\ker \varphi = \{ 0 \}.
$$

- If $2g = e_G$, then

$$
\varphi(2) = e_G,
$$

and the kernel contains 0 and 2:

\[

\ker \varphi = \{ 0, 2 \}.

\]

Summary for Example (b):

- The kernel depends on the order of \(g = \varphi(1) \).

- When \(g \) is the identity:

\[

\boxed{

\ker \varphi = \mathbb{Z}_3,

}

\]

meaning the homomorphism is trivial.

- When \(2g = e_G \) but \(g \neq e_G \):

\[

\boxed{

\ker \varphi = \{ 0, 2 \}.

}

\]

- When neither condition holds:

\[

\boxed{

\ker \varphi = \{ 0 \}.

}

\]

Additional Considerations

Homomorphisms from $\mathbb{Z}$ to Finite Cyclic Groups

Homomorphisms from $\mathbb{Z}$ to a finite cyclic group $\mathbb{Z}_n$ are entirely characterized by the image of 1, which must be an element of $\mathbb{Z}_n$. The kernel in such cases is always a subgroup of $\mathbb{Z}$, typically of the form:

$$
k \mathbb{Z} = \{ kn : n \in \mathbb{Z} \},
$$

where k divides n based on

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