Consider A Butterworth Lowpass Filter Of Order 3 And Cut-off Frequency (w/c) Of 1. (i) Derive The Filter

Consider A Butterworth Lowpass Filter Of Order 3 And Cut-off Frequency (w/c) Of 1. (i) Derive The Filter

---

Introduction

In the realm of electronic signal processing, filters play a crucial role in shaping and controlling the frequency content of signals. Among various types, the Butterworth filter is renowned for its maximally flat magnitude response in the passband, making it highly desirable for applications requiring minimal signal distortion within the passband.

This article delves into the derivation of a third-order Butterworth lowpass filter with a cutoff frequency (ω_c) of 1 rad/sec. Such a filter is fundamental in numerous practical applications, including audio processing, communication systems, and instrumentation, where a smooth and flat response is critical.

Understanding the underlying mathematics and design principles of Butterworth filters empowers engineers to create efficient and reliable filtering solutions tailored to specific frequency requirements.

Overview of Butterworth Filters

What Is a Butterworth Filter?

A Butterworth filter is a type of electronic filter designed to have as flat a frequency response as possible in the passband. Unlike other filters that might introduce ripples (e.g., Chebyshev or Elliptic filters), Butterworth filters aim for a smooth response, which is particularly advantageous in applications where signal integrity is paramount.

Key features of Butterworth filters:

- Maximal flatness in passband
- Gradual roll-off beyond cutoff frequency
- Predictable and stable performance

- Simple mathematical formulation

Why Choose a Butterworth Filter?

The primary reasons for selecting a Butterworth filter include:

- Flat and ripple-free passband
- Ease of design and implementation
- Suitable for applications requiring minimal distortion

Design Specifications

Before deriving the filter, it's essential to specify its key parameters:

- Order (n): 3 (third-order)
- Cutoff frequency (ω_c): 1 rad/sec
- Type: Lowpass filter

These parameters define the filter's complexity and frequency response characteristics.

Mathematical Foundations of Butterworth Filters

Magnitude Response

The magnitude squared response of an n-th order Butterworth lowpass filter is given by:

$$
|H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}
$$

where:
- ω is the angular frequency
- ω_c is the cutoff frequency
- n is the filter order

This formula indicates that at $\omega = \omega_c$, the magnitude response is $\frac{1}{\sqrt{2}}$, corresponding to the -3 dB point.

Poles of the Butterworth Filter

The poles of the Butterworth filter are equally spaced around a circle in the complex plane, lying on a circle of radius ω_c. For a normalized lowpass filter:

$$
s_k = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi (2k + 1)}{2n} \right)} \quad \text{for} \quad k=0,1,...,n-1
$$

For $n=3$, the poles are located at:

$$
s_k = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi (2k + 1)}{6} \right)}
$$

Deriving the Transfer Function of the 3rd-Order Butterworth Lowpass Filter

Step 1: Calculate the Poles

For $n=3$, the poles are at:

$$
k=0: \quad s_0 = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi \times 1}{6} \right)} = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi}{6} \right)} = \omega_c \cdot e^{j \left(\frac{2\pi}{3} \right)}
$$

$$
k=1: \quad s_1 = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi \times 3}{6} \right)} = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi}{2} \right)} = \omega_c \cdot e^{j \pi}
$$

$$
k=2: \quad s_2 = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{\pi \times 5}{6} \right)} = \omega_c \cdot e^{j \left(\frac{\pi}{2} + \frac{5\pi}{6} \right)} = \omega_c \cdot e^{j \left(\frac{4\pi}{3} \right)}
$$

Since $\omega_c = 1$, the poles are:

\[

$$
s_0 = e^{j 120^\circ} = -\frac{1}{2} + j \frac{\sqrt{3}}{2}
$$

$$
s_1 = e^{j 180^\circ} = -1 + j 0
$$

$$
s_2 = e^{j 240^\circ} = -\frac{1}{2} - j \frac{\sqrt{3}}{2}
$$

Note that for stability, the poles must lie in the left half-plane, which they do.

Note: For an analog lowpass filter, the transfer function is derived from these poles.

Step 2: Form the Transfer Function

The general form of the transfer function $H(s)$ for a Butterworth filter with these poles is:

$$
H(s) = \frac{1}{(s - s_0)(s - s_1)(s - s_2)}
$$

Expressed explicitly:

$$
H(s) = \frac{1}{(s - s_0)(s - s_1)(s - s_2)}
$$

Calculating the quadratic factors:

$$
(s - s_0)(s - s_2) = s^2 - (s_0 + s_2)s + s_0 s_2
$$

Since $s_0 = -\frac{1}{2} + j \frac{\sqrt{3}}{2}$ and $s_2 = -\frac{1}{2} - j \frac{\sqrt{3}}{2}$:

$$
s_0 + s_2 = -\frac{1}{2} + j \frac{\sqrt{3}}{2} - \frac{1}{2} - j \frac{\sqrt{3}}{2} = -1
$$

$$
s_0 s_2 = \left(-\frac{1}{2} + j \frac{\sqrt{3}}{2}\right) \left(-\frac{1}{2} - j \frac{\sqrt{3}}{2}\right)
$$

Multiplying out:

$$
= \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1
$$

Thus, the quadratic factor:

$$
s^2 + s + 1
$$

Including the pole at $s = -1$:

$$
H(s) = \frac{1}{(s + 1)(s^2 + s + 1)}
$$

Step 3: Normalize the Transfer Function

Since the cutoff frequency is 1 rad/sec, the transfer function is already normalized. To get the standard form, express $H(s)$ as:

$$
H(s) = \frac{\omega_c^n}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)}
$$

For $\omega_c = 1$:

$$
H(s) = \frac{1}{(s + 1)(s^2 + s + 1)}
$$

This is the normalized transfer function of the third-order Butterworth lowpass filter.

Implementation and Realization

Step 1: Derive the Circuit Components

The transfer function can be realized using standard filter topologies such as the Sallen-Key or Multiple Feedback (MFB) configurations. For a third-order lowpass Butterworth filter, the MFB topology is preferred due to its stability and ease of tuning.

Key components:

- Operational amplifiers
- Resistors
- Capacitors

Step 2: Component Value Calculation

The normalized transfer function coefficients guide the selection of resistor and capacitor values. For simplicity, choose standard capacitor values (e.g., 1 μF) and compute resistor values accordingly.

Frequently Asked Questions

What is a Butterworth lowpass filter and why is it preferred in signal processing?

A Butterworth lowpass filter is a type of filter characterized by a flat frequency response in the passband, meaning it has no ripples. It is preferred because it provides a smooth, monotonic response that effectively attenuates frequencies beyond the cutoff point without introducing ripples, making it ideal for applications requiring a flat passband.

How is the transfer function of a third-order Butterworth lowpass filter derived?

The transfer function of a third-order Butterworth lowpass filter is derived by ensuring its magnitude response satisfies $|H(j\omega)|^2 = 1 / (1 + (\omega/\omega_c)^{2n})$, where $n=3$ for a third-order filter. The poles are located on a circle in the left-half s-plane, equally spaced at 60° angles, leading to the specific pole locations used to formulate the transfer function.

What are the pole locations for a third-order Butterworth lowpass filter with cutoff frequency ω_c?

For a third-order Butterworth lowpass filter, the poles are located at: $s = \omega_c e^{j(\pi/2 + (2k+1)\pi/(2n))}$ for $k=0,1,2$. Specifically, the poles are at angles of 150°, 210°, and 330°, with magnitudes equal to the cutoff frequency ω_c, positioned in the left-half s-plane to ensure stability.

How do you derive the transfer function H(s) of the third-order Butterworth filter with cutoff frequency $\omega_c=1$?

To derive H(s), identify the poles on the left-half s-plane at angles of 150°, 210°, and 330° with magnitude 1. The transfer function is then expressed as $H(s) = K / [(s - s_1)(s - s_2)(s - s_3)]$, where

s_1, s_2, and s_3 are the pole locations. Normalizing for $\omega_c=1$ simplifies the expression, and K is chosen to ensure unity gain at zero frequency.

What is the significance of the cutoff frequency in the Butterworth lowpass filter, and how is it incorporated in the transfer function?

The cutoff frequency ω_c defines the boundary between the passband and stopband, where the filter begins to significantly attenuate signals. In the transfer function, ω_c scales the pole locations and determines the frequency at which the magnitude response drops to $\sqrt{2}/2$ (~0.707) of the passband gain, ensuring the filter's desired cutoff behavior.

Can you provide the explicit form of the transfer function H(s) for this third-order Butterworth lowpass filter with $\omega_c=1$?

Yes. The poles for $\omega_c=1$ are at $s = e^{j(\pi/2 + (2k+1)\pi/6)}$ for k=0,1,2. The transfer function is: $H(s) = 1 / [(s - s_1)(s - s_2)(s - s_3)]$. Substituting the pole values, the transfer function becomes a rational function with specific real and complex conjugate poles, providing a flat passband response up to the cutoff frequency.

What are practical considerations when designing a third-order Butterworth lowpass filter for real-world applications?

Practical considerations include component tolerances affecting filter accuracy, the choice of circuit topology (such as Sallen-Key or multiple feedback), stability, and the impact of non-idealities like parasitic elements. Ensuring precise cutoff frequency and flat response may require careful component selection and tuning, especially for high-frequency applications.

Additional Resources

Consider A Butterworth Lowpass Filter Of Order 3 And Cut-off Frequency (w/c) Of 1. (i) Derive The Filter

Designing filters is a fundamental task in signal processing, audio engineering, and communications systems. Among various filter types, the Butterworth lowpass filter is renowned for its maximally flat frequency response in the passband, making it an ideal choice when a smooth response without ripples is desired. In this article, we will undertake a detailed derivation of a Butterworth lowpass filter of order 3 with a cutoff frequency (w/c) of 1, providing an in-depth explanation suitable for students, engineers, or enthusiasts seeking to understand the theoretical foundations behind filter design.

Introduction to Butterworth Lowpass Filters

Before diving into the derivation, it’s essential to understand what characterizes a Butterworth lowpass filter:

- Maximally flat response: No ripples in the passband.
- Order (n): Determines the sharpness of the filter transition. Higher orders create a steeper roll-off.
- Cutoff frequency (w_c): The frequency at which the magnitude response drops to $1/\sqrt{2}$ of the passband gain (approximately -3 dB point).

For a 3rd order Butterworth lowpass filter with a cutoff frequency of w/c = 1, our goal is to derive the transfer function $H(s)$ that satisfies these specifications.

Key Concepts in Butterworth Filter Design

Poles and the Butterworth Polynomial

The defining characteristic of a Butterworth filter lies in its pole placement:

- The poles are equally spaced on a circle in the left-half of the s-plane.
- The transfer function is expressed as a ratio of polynomials, with the denominator polynomial known as the Butterworth polynomial.

Transfer Function Form

The standard form of the transfer function for an n^{th} order Butterworth lowpass filter is:

$$
H(s) = \frac{G_0}{B(s)}
$$

where:

- G_0 is the passband gain (often normalized to 1).
- $B(s)$ is the Butterworth polynomial, which is the product of the factors corresponding to each pole.

Step 1: Determine the Poles of the Filter

Calculating the Poles

For an order 3 Butterworth filter, the poles are located at:

$$
s_k = w_c \cdot e^{j\left(\frac{\pi}{2} + \frac{\pi (2k + 1)}{2n} \right)}, \quad k=0,1,\dots,n-1
$$

Since $w_c=1$, the poles are positioned at:

$$
s_k = e^{j\left(\frac{\pi}{2} + \frac{\pi (2k + 1)}{2 \times 3} \right)} = e^{j\left(\frac{\pi}{2} + \frac{\pi (2k + 1)}{6} \right)}
$$

\]

Calculating for each \(k\):

- For \(k=0\):

\[
s_0 = e^{j\left(\frac{\pi}{2} + \frac{\pi (1)}{6} \right)} = e^{j\left(\frac{\pi}{2} + \frac{\pi}{6} \right)} = e^{j\left(\frac{2\pi}{3} \right)}
\]

- For \(k=1\):

\[
s_1 = e^{j\left(\frac{\pi}{2} + \frac{\pi (3)}{6} \right)} = e^{j\left(\frac{\pi}{2} + \frac{\pi}{2} \right)} = e^{j\pi}
\]

- For \(k=2\):

\[
s_2 = e^{j\left(\frac{\pi}{2} + \frac{\pi (5)}{6} \right)} = e^{j\left(\frac{\pi}{2} + \frac{5\pi}{6} \right)} = e^{j\left(\frac{4\pi}{3} \right)}
\]

Expressing these in rectangular form:

- \(s_0 = e^{j 120^\circ} = -\frac{1}{2} + j \frac{\sqrt{3}}{2}\)
- \(s_1 = e^{j 180^\circ} = -1 + j 0\)
- \(s_2 = e^{j 240^\circ} = -\frac{1}{2} - j \frac{\sqrt{3}}{2}\)

Since the Butterworth poles are symmetric with respect to the real axis, the poles in the left-half s-plane are:

\[
s_{p1} = -\frac{1}{2} + j \frac{\sqrt{3}}{2}
\]
\[
s_{p2} = -\frac{1}{2} - j \frac{\sqrt{3}}{2}
\]
\[
s_{p3} = -1
\]

(Note: The pole at \(-1\) is real and in the left-half plane, ensuring stability.)

Step 2: Form the Butterworth Polynomial

The denominator of the transfer function, \(B(s)\), is constructed from these poles:

$$
B(s) = (s - s_{p1})(s - s_{p2})(s - s_{p3})
$$

Substituting the poles:

$$
B(s) = (s - [-\frac{1}{2} + j \frac{\sqrt{3}}{2}]) (s - [-\frac{1}{2} - j \frac{\sqrt{3}}{2}]) (s + 1)
$$

Simplify each factor:

$$
B(s) = (s + \frac{1}{2} - j \frac{\sqrt{3}}{2})(s + \frac{1}{2} + j \frac{\sqrt{3}}{2})(s + 1)
$$

Step 2.1: Multiply the complex conjugate pair

$$
(s + \frac{1}{2} - j \frac{\sqrt{3}}{2})(s + \frac{1}{2} + j \frac{\sqrt{3}}{2}) = \left(s + \frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2
$$

Calculating:

$$
= \left(s + \frac{1}{2} \right)^2 + \frac{3}{4}
$$

$$
= s^2 + s + \frac{1}{4} + \frac{3}{4} = s^2 + s + 1
$$

Step 2.2: Multiply by the real pole at -1:

$$
B(s) = (s^2 + s + 1)(s + 1)
$$

Expand:

$$
B(s) = s^3 + s^2 + s + s^2 + s + 1 = s^3 + 2s^2 + 2s + 1
$$

Thus, the denominator polynomial is:

$$
B(s) = s^3 + 2s^2 + 2s + 1
$$

Step 3: Normalization and Final Transfer Function

3.1: Normalize the gain

For a unity-gain lowpass filter, the numerator is set so that the magnitude at $s=0$ (DC) is 1:

$$
H(s) = \frac{G_0}{B(s)} = \frac{1}{s^3 + 2s^2 + 2s + 1}
$$

3.2: Express the transfer function

The final normalized transfer function of the third-order Butterworth lowpass filter with cutoff frequency $\omega_c=1$ is:

$$
H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
$$

Step 4: Frequency Response and Characteristics

4.1: Magnitude response

The magnitude response $|H(j\omega)|$ indicates how the filter attenuates signals at different frequencies. For this filter:

$$
H(j\omega) = \frac{1}{(j\omega)^3 + 2(j\omega)^2 + 2(j\omega) + 1}
$$

Simplify the denominator:

$$
(j\omega)^3 = j^3 \omega^3 = -j \omega^3
$$
$$
2(j\omega)^2 = 2 (j^2 \omega^2) = 2(-1) \omega^2 = -2 \omega^2
$$
$$
2 (j\omega) = 2j \omega
$$

So:

$$
D(j\omega) = -j \omega^3 - 2 \omega^2 +
$$

[Consider A Butterworth Lowpass Filter Of Order 3 And Cut Off Frequency W C Of 1 I Derive The Filter]

Consider A Butterworth Lowpass Filter Of Order 3 And Cut Off Frequency W C Of 1 I Derive The Filter

Related Articles

- Cheryl Led The Fundraisers That Allowed The Theater To Be Renovated, But She Did Not Brag About Her Good
- Good Accounting Procedures Require That A Copy Of The Accounting System Be Maintained. Where Should This
- Animations In Which One Of The Following Categories Cause Objects To Change While They Are On A Slide?A)

Back to Home