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When approaching the problem of integrating a function such as F(x) = 1 + e^{-x} \cos(4x) over the interval [0,1], it is essential to explore various methods of integration to achieve an accurate and efficient solution. This article will delve into multiple techniques, including analytical methods, approximation strategies, and numerical integration, providing a comprehensive understanding of how to evaluate the integral of this specific function.

Understanding the Function F(x) = 1 + e^{-x} \cos(4x)

Before applying different integration methods, it is crucial to analyze the structure of the given function.

Components of the Function

The function comprises two parts:

- A constant term: 1
- A product of exponential decay and cosine oscillation: e^{-x} \cos(4x)

This combination suggests that the integral will involve both straightforward and more complex components, especially due to the oscillatory nature of cosine and the exponential decay.

Relevance of the Components

- The constant term's integral over [0,1] is simple.
- The exponential-cosine term requires more careful handling, possibly involving integration techniques suited for oscillatory functions.

Analytical Integration Techniques

Analytical methods involve deriving an exact expression for the integral, which can then be evaluated at the bounds. For F(x), the integral over [0,1] is:

$$
I = \int_{0}^{1} \left(1 + e^{-x} \cos(4x) \right) dx
$$

which separates into:

$$
I = \int_{0}^{1} 1\, dx + \int_{0}^{1} e^{-x} \cos(4x) dx
$$

Integral of the Constant Term

$$
\int_{0}^{1} 1\, dx = 1
$$

This is straightforward.

Integral of the Exponential-Cosine Term

The key challenge lies in evaluating:

$$
J = \int_{0}^{1} e^{-x} \cos(4x) dx
$$

This integral is a standard form that can be solved using the method of complex exponentials or by applying integration by parts.

Method 1: Using Complex Exponentials

Recall that:

$$
\cos(4x) = \Re \left(e^{i4x} \right)
$$

Thus:

$$
J = \Re \left(\int_{0}^{1} e^{-x} e^{i4x} dx \right) = \Re \left(\int_{0}^{1} e^{(-1 + i4)x} dx \right)
$$

Evaluating the integral:

$$
\int e^{ax} dx = \frac{e^{ax}}{a}
$$

where $a = -1 + i4$. Therefore,

$$
J = \Re \left(\frac{e^{a x}}{a} \bigg|_{0}^{1} \right) = \Re \left(\frac{e^{a} - 1}{a} \right)
$$

Calculating $e^{a} = e^{-1 + i4} = e^{-1} e^{i4}$.

- e^{-1} is a real number.
- $e^{i4} = \cos 4 + i \sin 4$.

Thus:

$$
e^{a} = e^{-1} (\cos 4 + i \sin 4)
$$

Putting it all together:

$$
J = \Re \left(\frac{e^{-1} (\cos 4 + i \sin 4) - 1}{-1 + i4} \right)
$$

To find the real part, rationalize the denominator:

$$
a = -1 + i4
$$
$$
|a|^2 = (-1)^2 + (4)^2 = 1 + 16 = 17
$$

Multiply numerator and denominator by the complex conjugate of a:

$$
\frac{e^{a} - 1}{a} \times \frac{-1 - i4}{-1 - i4} = \frac{(e^{a} - 1)(-1 - i4)}{|a|^2}
$$

Calculating numerator:

$$
N = (e^{a} - 1)(-1 - i4)
$$

Express $e^{a} - 1$ as:

\[

$$
e^{-1} (\cos 4 + i \sin 4) - 1
$$

Now, multiply out:

$$
N = \left[e^{-1} (\cos 4 + i \sin 4) - 1 \right] \times (-1 - i4)
$$

This expansion yields real and imaginary parts, which, after simplification, will give the value of J. The detailed algebra can be carried out step-by-step or computed numerically for accuracy.

Method 2: Integration by Parts

Alternatively, the integral:

$$
J = \int e^{-x} \cos(4x) dx
$$

can be approached via integration by parts twice, or using the standard integral:

$$
\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C
$$

where $a = -1$, $b=4$.

Plugging in:

$$
J = \frac{e^{-x}}{(-1)^2 + 4^2} \left(-1 \cos 4x + 4 \sin 4x \right) + C = \frac{e^{-x}}{1 + 16} (-\cos 4x + 4 \sin 4x) + C
$$

$$
J = \frac{e^{-x}}{17} (- \cos 4x + 4 \sin 4x) + C
$$

Evaluating from 0 to 1:

$$
J = \left[\frac{e^{-x}}{17} (- \cos 4x + 4 \sin 4x) \right]_0^1
$$

Compute:

At $x=1$:

$$
J_1 = \frac{e^{-1}}{17} (- \cos 4 + 4 \sin 4)
$$

At $x=0$:

$$
J_0 = \frac{1}{17} (-1 + 0) = - \frac{1}{17}
$$

So,

$$
J = J_1 - J_0 = \frac{e^{-1}}{17} (- \cos 4 + 4 \sin 4) + \frac{1}{17}
$$

Now, the total integral:

$$
I = 1 + J = 1 + \frac{e^{-1}}{17} (- \cos 4 + 4 \sin 4) + \frac{1}{17}
$$

which simplifies to:

$$
I = 1 + \frac{1}{17} + \frac{e^{-1}}{17} (- \cos 4 + 4 \sin 4)
$$

This expression provides the exact value of the definite integral over [0,1].

Numerical Integration Methods

While analytical solutions are precise, sometimes functions are too complex or lack elementary antiderivatives, necessitating numerical approaches.

Common Numerical Techniques

- Trapezoidal Rule
- Simpson’s Rule
- Gaussian Quadrature
- Romberg Integration

Implementing Numerical Methods for F(x)

To approximate the integral of F(x) over [0,1], follow these steps:

1. Partition the interval into subintervals (e.g., 10, 50, 100 divisions).
2. Evaluate the function at each node.
3. Apply the chosen rule to compute the approximate integral.

Example: Simpson’s Rule

Given n even subintervals:

$$
\int_{a}^{b} f(x) dx \approx \frac{h}{3} \left[f(x_0) + 4 \sum_{i=1,3,5,\ldots}^{n-1} f(x_i) + 2 \sum_{i=2,4,6,\ldots}^{n-2} f(x_i) + f(x_n) \right]
$$

where $h = \frac{b - a}{n}$.

Applying this to F(x):

- Calculate $f(x_i) = 1 + e^{-x_i} \cos(4x_i)$ for each x_i.
- Sum accordingly.
- Increase n for higher accuracy.

Applying Various Strategies: Comparative Analysis

Understanding the strengths and limitations of each method helps in selecting the most suitable approach.

Analytical Methods