Consider The Following Function. F(x)=xln8x,a=1,n=3,0.7x1.3(a) Approximate F By A Taylor Polynomial With

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Introduction

In mathematical analysis, Taylor polynomials are essential tools for approximating functions near a specific point. They allow us to express complicated functions as polynomials, which are easier to evaluate and analyze. In this article, we focus on the function $F(x) = x \ln(8x)$, with a particular interest in approximating it near $a=1$ using a third-degree Taylor polynomial. The parameters $a=1$ and $n=3$ indicate that we will construct a third-order Taylor polynomial centered at $x=1$. Additionally, the expression involving $0.7 \times 1.3(a)$ suggests a potential domain or point of interest, but the primary goal remains deriving the Taylor polynomial for $F(x)$ around $a=1$.

Understanding the Function $F(x) = x \ln(8x)$

Basic properties and domain

The function $F(x) = x \ln(8x)$ combines a linear term x with a logarithmic term $\ln(8x)$. Some key points include:

- Domain: Since the natural logarithm $\ln(8x)$ is defined for $8x > 0$, the domain of $F(x)$ is $x > 0$.
- Behavior near zero: As $x \to 0^+$, $\ln(8x) \to -\infty$, and hence $F(x)$ tends to zero from below or above depending on the rate.
- Behavior at $x=1$: The point of expansion $a=1$ falls well within the domain, making it suitable for Taylor expansion.

Rewriting $F(x)$ for easier differentiation

To facilitate derivatives, rewrite:

$$
F(x) = x \ln(8x) = x (\ln 8 + \ln x) = x \ln 8 + x \ln x
$$

This separation simplifies derivatives as:

- $x \ln 8$ is linear in x.
- $x \ln x$ involves the product of x and $\ln x$, which has well-known derivatives.

Calculating Derivatives at $a=1$

To construct the Taylor polynomial of degree 3 centered at $a=1$, we need $F(a)$, $F'(a)$, $F''(a)$, and $F'''(a)$.

First derivative $F'(x)$

Using the product rule:

$$
F(x) = x \ln(8x) = x \left(\ln 8 + \ln x\right)
$$

Derivative:

$$
F'(x) = \frac{d}{dx} \left[x \ln 8 + x \ln x \right] = \ln 8 + \frac{d}{dx} (x \ln x)
$$

Recall:

$$
\frac{d}{dx} (x \ln x) = \ln x + 1
$$

Therefore:

$$
F'(x) = \ln 8 + \ln x + 1

\]

Evaluate at \(x=1\):

\[
F'(1) = \ln 8 + \ln 1 + 1 = \ln 8 + 0 + 1
\]

Recall \(\ln 8 = \ln 2^3 = 3 \ln 2 \approx 3 \times 0.6931 = 2.0794\):

\[
F'(1) \approx 2.0794 + 1 = 3.0794
\]

Second derivative \(F''(x)\)

Differentiate \(F'(x)\):

\[
F'(x) = \ln 8 + \ln x + 1
\]

\[
F''(x) = \frac{d}{dx} (\ln 8 + \ln x + 1) = \frac{1}{x}
\]

At \(x=1\):

\[
F''(1) = \frac{1}{1} = 1
\]

Third derivative \(F'''(x)\)

Differentiate \(F''(x)\):

\[
F'''(x) = -\frac{1}{x^2}
\]

At \(x=1\):

\[
F'''(1) = -1
\]

Constructing the Taylor Polynomial of Degree 3 at $a=1$

The general form of the Taylor polynomial centered at $a=1$ is:

$$
T_3(x) = F(1) + F'(1)(x-1) + \frac{F''(1)}{2!}(x-1)^2 + \frac{F'''(1)}{3!}(x-1)^3
$$

Calculate $F(1)$:

$$
F(1) = 1 \times \ln(8 \times 1) = \ln 8 \approx 2.0794
$$

Now, write the polynomial:

$$
T_3(x) = 2.0794 + 3.0794 (x-1) + \frac{1}{2} (x-1)^2 - \frac{1}{6} (x-1)^3
$$

Final Approximate Formula

The third-degree Taylor polynomial approximation of $F(x)$ near $x=1$ is:

$$
\boxed{
F(x) \approx 2.0794 + 3.0794 (x-1) + \frac{1}{2} (x-1)^2 - \frac{1}{6} (x-1)^3
}
$$

This polynomial provides a close approximation of $F(x)$ for x near 1.

Analyzing the Approximation

Range of validity

- Taylor polynomials are most accurate near the center point $a=1$.
- The error depends on higher derivatives and increases as x moves further from 1.
- For practical purposes, the approximation is reliable within a small interval around 1 (e.g., $0.8 \leq x \leq 1.2$).

Using the approximation

- To estimate $F(x)$ at some point near 1, substitute into $T_3(x)$.
- This is particularly useful when direct computation of $F(x)$ is complex or computationally expensive.

Conclusion

Constructing Taylor polynomials is a powerful method for approximating complex functions, especially around a point of interest. For the function $F(x) = x \ln(8x)$, centered at $a=1$, the third-degree Taylor polynomial provides a practical approximation that captures the behavior of $F(x)$ near $x=1$. The derivatives calculated at $a=1$ reveal the local properties of the function, such as the slope, concavity, and rate of change. This approach is fundamental in various fields, including numerical analysis, physics, and engineering, where approximations enable simpler computations and insights into the function's behavior.

References and Further Reading

- Stewart, J. (2015). Calculus: Early Transcendentals. Cengage Learning.
- Thomas, G. B., & Finney, R. L. (2000). Calculus and Analytic Geometry. Pearson.
- Online resources: Wolfram Alpha, Khan Academy, MIT OpenCourseWare on Taylor series and approximation techniques.

Frequently Asked Questions

What is the function F(x) given in the problem?

The function is $F(x) = x \ln(8x)$.

What are the values of a and n specified in the problem?

The given values are $a = 1$ and $n = 3$.

What is the goal when approximating F(x) with a Taylor polynomial?

The goal is to find a polynomial of degree 3 that approximates $F(x)$ near $x = 1$.

How do you find the Taylor polynomial of degree 3 centered at a = 1?

You compute the derivatives of F at $x = 1$ up to the third order and construct the polynomial: $P_3(x) = F(1) + F'(1)(x-1) + (F''(1)/2!)(x-1)^2 + (F'''(1)/3!)(x-1)^3$.

What is the first step in approximating F(x) with a Taylor polynomial?

Calculate $F(1)$ and the derivatives $F'(x)$, $F''(x)$, and $F'''(x)$, then evaluate each at $x = 1$.

How do you compute the derivatives of F(x) = x ln(8x)?

Use product rule and chain rule: $F'(x) = \ln(8x) + 1$, $F''(x) = 1/x$, and $F'''(x) = -1/x^2$.

What is the approximate Taylor polynomial of F(x) at x = 1?

Calculating the derivatives at 1: $F(1) = 1 \ln(8) = \ln(8)$, $F'(1) = \ln(8) + 1$, $F''(1) = 1$, $F'''(1) = -1$. Plugging into the polynomial: $P_3(x) = \ln(8) + (\ln(8)+1)(x-1) + (1/2)(x-1)^2 + (-1/6)(x-1)^3$.

Why is using a Taylor polynomial useful for approximating F(x)?

It provides a polynomial approximation of $F(x)$ near a point, making calculations easier and faster, especially for values close to the center $a = 1$.

Additional Resources

Consider The Following Function. F(x)=xln8x, a=1, n=3, 0.7x1.3(a) Approximate F By A Taylor Polynomial With

When analyzing complex functions, the Taylor polynomial becomes an invaluable tool, offering a means to approximate functions near a specific point with a polynomial of finite degree. In this review, we delve into the function $F(x) = x \ln(8x)$, examining how to approximate it around the point $a=1$ using a third-degree Taylor polynomial. We will explore the theoretical foundations, practical applications, and the nuances involved in this approximation process, providing a comprehensive guide for students, educators, and practitioners alike.

Understanding the Function $F(x) = x \ln(8x)$

Before diving into the approximation process, it’s crucial to understand the behavior and properties of the given function.

Function Characteristics

- Domain: The function is defined for $x > 0$, since $\ln(8x)$ requires positive arguments.
- Continuity & Differentiability: $F(x)$ is continuous and infinitely differentiable for all $x > 0$, making it suitable for Taylor series expansion.
- Behavior near $a=1$: At $x=1$, $F(1) = 1 \times \ln(8) \approx 2.0794$.

Rewriting the Function

Expressing $F(x)$ in a form conducive to differentiation:

$$
F(x) = x \ln(8x) = x (\ln 8 + \ln x) = x \ln 8 + x \ln x
$$

This decomposition simplifies derivatives and helps in understanding the function's local behavior.

Why Approximate $F(x)$ with a Taylor Polynomial?

Taylor polynomial approximation provides:
- Simplification: Replacing complex functions with polynomials simplifies calculations.
- Local accuracy: Near the point $a=1$, the polynomial closely matches the function.
- Analytical insights: Derivatives inform about the function's curvature, increasing understanding.

Applications include:
- Numerical methods
- Optimization problems
- Integrations and differential equations where exact solutions are complex or unknown.

Constructing the Taylor Polynomial at $a=1$

The general form of the Taylor polynomial of degree n about a is:
$$
T_n(x) = \sum_{k=0}^n \frac{F^{(k)}(a)}{k!} (x - a)^k
$$
where $F^{(k)}(a)$ is the k^th derivative evaluated at a.

Given $n=3$ and $a=1$, our task involves calculating derivatives up to third order at $x=1$.

Step 1: Calculate $F(1)$

$$
F(1) = 1 \times \ln(8 \times 1) = \ln 8 \approx 2.0794
$$

Step 2: Compute derivatives $F'(x), F''(x), F'''(x)$

Recall:
$$
F(x) = x \ln(8x)
$$

Expressed as:

$$
F(x) = x \ln 8 + x \ln x
$$

Derivative calculations:

- First derivative:

$$
F'(x) = \frac{d}{dx} (x \ln 8 + x \ln x) = 0 + \ln x + 1
$$

because:

$$
\frac{d}{dx} (x \ln x) = \ln x + 1
$$

- Evaluate at $x=1$:

$$
F'(1) = \ln 1 + 1 = 0 + 1 = 1
$$

- Second derivative:

$$
F''(x) = \frac{d}{dx} (\ln x + 1) = \frac{1}{x}
$$

- Evaluate at $x=1$:

$$
F''(1) = 1
$$

- Third derivative:

$$
F'''(x) = - \frac{1}{x^2}
$$

- Evaluate at $x=1$:

$$
F'''(1) = -1
$$

Forming the Third-Degree Taylor Polynomial $T_3(x)$

Using the derivatives at $a=1$:

$$
T_3(x) = F(1) + F'(1)(x - 1) + \frac{F''(1)}{2!}(x - 1)^2 + \frac{F'''(1)}{3!}(x - 1)^3
$$

Substituting the values:

$$
T_3(x) = \ln 8 + 1 \times (x - 1) + \frac{1}{2} (x - 1)^2 - \frac{1}{6} (x - 1)^3
$$

Expressed explicitly:

$$
T_3(x) = \ln 8 + (x - 1) + \frac{1}{2}(x - 1)^2 - \frac{1}{6}(x - 1)^3
$$

This polynomial approximates $F(x)$ near $x=1$.

Analyzing the Approximation: Features & Limitations

Features:
- Accuracy near $a=1$: The polynomial provides a good approximation within a small neighborhood around $x=1$.
- Ease of computation: Polynomial expressions are easier to evaluate and manipulate.
- Derivative insights: The polynomial captures the local behavior of the function, including slope and concavity.

Limitations:
- Limited radius of convergence: The approximation becomes less accurate as x moves farther from 1.
- Higher-order effects ignored: For more precise modeling, higher-degree polynomials or other methods are required.
- Possible divergence: For points where x approaches 0 (where $\ln x$ diverges), the approximation fails.

Practical Applications and Usage Scenarios

- Numerical analysis: Estimating function values without complex calculations.
- Optimization: Finding local maxima or minima via derivative approximations.
- Economics and physics: Modeling functions that resemble $x \ln x$ in entropy, information theory, or thermodynamics.
- Educational purposes: Demonstrating how derivatives inform about local behavior and how Taylor series work.

Pros and Cons Summary

Pros:
- Simplifies complex functions into manageable polynomials.
- Provides insight into local behavior via derivatives.
- Useful in numerical methods and computational algorithms.

Cons:
- Limited accuracy for points far from the expansion point.
- Derivation becomes cumbersome for very high-degree polynomials.
- Not suitable for functions with singularities or discontinuities near the point of expansion.

Conclusion

Approximating \(F(x) = x \ln(8x) \) using a third-degree Taylor polynomial centered at \(a=1 \) offers a practical approach to understanding and simplifying the behavior of the function around that point. The process involves calculating derivatives, constructing the polynomial, and analyzing its features and limitations. While the approximation is powerful for local analysis, users must be cautious about its range of validity. Overall, Taylor series serve as a cornerstone in mathematical analysis, bridging the gap between complex functions and their manageable polynomial counterparts, enabling deeper insights and efficient computations across various scientific disciplines.

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