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Consider The Planar Linear System X' = AX, Where [2-3] A = 3 2 (a) Find The General Solution. (b) Sketch

Understanding planar linear systems is fundamental in differential equations and dynamical systems, as they serve as models for various phenomena in physics, engineering, biology, and economics. The system $\mathbf{X'} = A \mathbf{X}$ with a constant matrix A is particularly important because its solutions can be characterized analytically through eigenvalues and eigenvectors. This article provides a comprehensive guide to analyzing such systems, specifically focusing on the example where

$$
A = \begin{bmatrix}
2 & -3 \\
3 & 2
\end{bmatrix}
$$

and aims to answer two key questions:

1. Find the general solution of the system.
2. Sketch the phase portrait to visualize the behavior of solutions.

Introduction to Planar Linear Systems

A planar linear system is written in matrix form as:

$$
\mathbf{X'} = A \mathbf{X}
$$

where:

- $\mathbf{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is the state vector,
- A is a 2×2 constant matrix.

The goal is to find the general solution that describes the evolution of $\mathbf{X}(t)$ over time, given initial conditions.

Part (a): Finding the General Solution

Step 1: Write down the matrix A

Given:

$$
A = \begin{bmatrix}
2 & -3 \\
3 & 2
\end{bmatrix}
$$

This matrix governs the dynamics of the system.

Step 2: Find the Eigenvalues of A

Eigenvalues determine the qualitative nature of solutions—they tell us whether solutions tend to infinity, decay, or oscillate.

To find eigenvalues λ, solve:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix:

$$
A - \lambda I = \begin{bmatrix}
2 - \lambda & -3 \\
3 & 2 - \lambda
\end{bmatrix}
$$

Calculate the determinant:

$$
(2 - \lambda)(2 - \lambda) - (-3)(3) = (2 - \lambda)^2 + 9
$$

Set equal to zero:

\[
(2 - \lambda)^2 + 9 = 0
\]

Simplify:

\[
(2 - \lambda)^2 = -9
\]

\[
(2 - \lambda) = \pm 3i
\]

Solve for \(\lambda \):

\[
\lambda = 2 \mp 3i
\]

Eigenvalues:

\[
\boxed{
\lambda_1 = 2 + 3i, \quad \lambda_2 = 2 - 3i
}
\]

Step 3: Find Eigenvectors

Choose \(\lambda = 2 + 3i \). Solve:

\[
(A - \lambda I) \mathbf{v} = 0
\]

which gives:

\[
\begin{bmatrix}
2 - (2 + 3i) & -3 \\
3 & 2 - (2 + 3i)
\end{bmatrix}
= \begin{bmatrix}
-3i & -3 \\
3 & -3i
\end{bmatrix}

\]

Set up the equations:

$$
-3i v_1 - 3 v_2 = 0
$$
$$
3 v_1 - 3i v_2 = 0
$$

From the first:

$$
-3i v_1 = 3 v_2 \Rightarrow v_2 = -i v_1
$$

Choose $v_1 = 1$, then:

$$
v_2 = -i
$$

Thus, an eigenvector corresponding to $\lambda = 2 + 3i$:

$$
\mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}
$$

Similarly, for $\lambda = 2 - 3i$, the eigenvector is the complex conjugate:

$$
\mathbf{\overline{v}} = \begin{bmatrix} 1 \\ i \end{bmatrix}
$$

Step 4: Write the General Solution

Using eigenvalues and eigenvectors, the general solution in complex form is:

$$
\mathbf{X}(t) = c_1 e^{(2 + 3i)t} \mathbf{v} + c_2 e^{(2 - 3i)t} \overline{\mathbf{v}}
$$

Expressed explicitly:

$$
\mathbf{X}(t) = e^{2t} \left[c_1 e^{3i t} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-3i t} \begin{bmatrix} 1 \\ i \end{bmatrix} \right]
$$

Using Euler's formula:

$$
e^{i \theta} = \cos \theta + i \sin \theta
$$

we can rewrite the solution as:

$$
\mathbf{X}(t) = e^{2t} \left[\alpha \cos 3t + \beta \sin 3t, \quad \text{with} \quad \alpha, \beta \in \mathbb{R} \right]
$$

which leads to the real-valued general solution:

$$
\boxed{
\mathbf{X}(t) = e^{2t} \left[\mathbf{C}_1 \cos 3t + \mathbf{C}_2 \sin 3t \right]
}
$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are vectors determined by initial conditions.

Part (b): Sketching the Phase Portrait

The phase portrait provides a visual understanding of the system's behavior over time, illustrating trajectories, stability, and type of equilibrium points.

Step 1: Identify the Nature of the Equilibrium Point

- The eigenvalues are complex with positive real parts ($\operatorname{Re}(\lambda) = 2 > 0$), indicating an unstable focus or spiral outward.

- Since the real part is positive, solutions spiral away from the origin as $t \to \infty$.

Step 2: Understand Trajectory Behavior

- The exponential term e^{2t} causes solutions to grow exponentially.

- The sine and cosine functions introduce oscillations, leading to spiral trajectories.

- The system's trajectories are outward spirals, moving away from the equilibrium at the origin.

Step 3: Sketching the Phase Portrait

To sketch the phase portrait:

1. Draw the coordinate axes representing x and y.

2. Mark the equilibrium point at the origin.

3. Illustrate a few trajectories starting near the origin, spiraling outward.

4. Indicate the directions of motion along the trajectories, which move away from the origin due to the positive real part of eigenvalues.

5. Add arrowheads to show the flow direction.

6. Optionally, include some sample solutions for different initial conditions to demonstrate the spiral nature.

Additional Insights:

- The qualitative behavior remains the same regardless of initial conditions, due to the linearity of the system.

- The amplitude of oscillations increases exponentially over time.

- The system exhibits an unstable focus, meaning it is not stable and solutions diverge away from the equilibrium.

Summary and Key Takeaways

- The matrix A with complex conjugate eigenvalues $2 \pm 3i$ leads to solutions characterized by exponential growth combined with oscillations.

- The general solution can be expressed in real form using sine and cosine functions multiplied by an exponential:

$$
\mathbf{X}(t) = e^{2t} \left[\mathbf{A} \cos 3t + \mathbf{B} \sin 3t \right]
$$

where $\mathbf{A}$ and $\mathbf{B}$ depend on initial conditions.

- The phase portrait indicates an unstable spiral that moves outward from the origin, showcasing the system's instability.

- Such analysis is crucial for predicting long-term behavior in systems modeled by linear differential equations, especially in control systems, physics, and other applied sciences.

Final Remarks

Understanding the eigenstructure of the matrix A is vital in analyzing the qualitative and quantitative behavior of planar linear systems. The method demonstrated—finding eigenvalues, eigenvectors, and rewriting solutions in real form—is standard and powerful. Visualizing the phase portrait complements the algebraic solution, providing intuition about system stability and dynamics. Whether for academic purposes or real-world modeling, mastering these techniques is essential for

Frequently Asked Questions

What is the general method to find the solution of the planar linear system X' = AX where A is a 2x2 matrix?

The general method involves finding the eigenvalues and eigenvectors of matrix A. The solutions are then expressed as linear combinations of exponential functions involving the eigenvalues multiplied by their corresponding eigenvectors.

Given the matrix A = [[3, 2], [a, 3]], how do you determine its eigenvalues for the system X' = AX?

Eigenvalues are found by solving the characteristic equation $\det(A - \lambda I) = 0$. For this matrix, it becomes $(3 - \lambda)^2 - 2a = 0$, leading to λ values that depend on the parameter a.

How do you find the general solution for the system X' = AX with A = [[3, 2], [a, 3]]?

First, compute eigenvalues λ; then find the corresponding eigenvectors. The general solution is a linear combination of terms like $e^{\lambda t}$ times the eigenvectors, adjusted for whether eigenvalues are real or complex.

What are the steps to sketch the phase portrait of the system X' = AX given the matrix A?

Identify eigenvalues and eigenvectors, determine stability (nodes, saddles, spirals), plot eigenvectors, and sketch trajectories showing the flow direction, considering whether eigenvalues are real or complex, positive or negative.

For A = [[3, 2], [a, 3]], what impact does changing the parameter 'a' have on the nature of the system's solutions and phase portrait?

Changing 'a' affects the eigenvalues; for certain values, the system may have real distinct eigenvalues, indicating node or saddle behavior, while for others, complex eigenvalues lead to spiral trajectories, influencing stability and the shape of the phase portrait.

Additional Resources

Linear Systems Analysis

Introduction to Planar Linear Systems

Imagine navigating a complex network of pathways where each route depends on your current position and the directions you choose. Similarly, in the realm of differential equations, linear systems serve as foundational tools for understanding the dynamic behavior of multiple interdependent variables. Among these, the planar linear system $\mathbf{X'} = A \mathbf{X}$ — where $\mathbf{X}$ is a vector in $\mathbb{R}^2$ and A is a constant 2×2 matrix — is perhaps the most accessible yet profoundly insightful model.

In this article, we will delve into the classic example of such a system, where the matrix A is given by:

\[

A = \begin{bmatrix}
2 & 3 \\
2 & 0
\end{bmatrix}
\]

Our goal is to analyze this system comprehensively by:

- Finding its general solution
- Sketching the phase portrait to understand the system's qualitative behavior

This approach not only exemplifies core techniques in differential equations but also offers a window into the rich dynamics that linear systems can exhibit.

Understanding the System: The Mathematical Framework

Before diving into calculations, let's establish what the system \(\mathbf{X'} = A \mathbf{X} \) entails.

- State variables: \(x(t) \) and \(y(t) \)
- System equations:

\[
\begin{cases}
x' = 2x + 3y \\
y' = 2x + 0 \cdot y
\end{cases}
\]

Expressed in matrix form:

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix}
= \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

The behavior of solutions hinges on the properties of matrix \(A\). To unlock these properties, the eigenvalues and eigenvectors of \(A\) are paramount.

Part (a): Finding the General Solution

Step 1: Determine Eigenvalues of \(A\)

Eigenvalues \(\lambda \) satisfy the characteristic equation:

\[
\det(A - \lambda I) = 0
\]

Calculating:

\[
A - \lambda I = \begin{bmatrix}
2 - \lambda & 3 \\
2 & -\lambda
\end{bmatrix}
\]

The determinant:

\[
(2 - \lambda)(-\lambda) - (3)(2) = 0
\]

Simplify:

\[
- (2 - \lambda) \lambda - 6 = 0
\]

\[
- (2\lambda - \lambda^2) - 6 = 0
\]

\[
- 2\lambda + \lambda^2 - 6 = 0
\]

Rearranged as:

\[
\lambda^2 - 2\lambda - 6 = 0
\]

Applying quadratic formula:

\[

\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-6)}}{2}
\]

\[
\lambda = \frac{2 \pm \sqrt{4 + 24}}{2} = \frac{2 \pm \sqrt{28}}{2}
\]

\[
\lambda = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}
\]

Eigenvalues:

\[
\boxed{
\lambda_1 = 1 + \sqrt{7} \quad \text{and} \quad \lambda_2 = 1 - \sqrt{7}
}
\]

These are real and distinct, indicating the system's solutions will involve exponential functions with different growth or decay rates.

Step 2: Find Eigenvectors

For each eigenvalue, solve \((A - \lambda I) \mathbf{v} = 0 \).

Eigenvector for \(\lambda_1 = 1 + \sqrt{7} \):

\[
A - \lambda_1 I = \begin{bmatrix}
2 - (1 + \sqrt{7}) & 3 \\
2 & - (1 + \sqrt{7})
\end{bmatrix}
= \begin{bmatrix}
1 - \sqrt{7} & 3 \\
2 & -1 - \sqrt{7}
\end{bmatrix}
\]

Solve for \(\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \):

From the first row:

\[
(1 - \sqrt{7}) v_1 + 3 v_2 = 0 \Rightarrow v_2 = - \frac{1 - \sqrt{7}}{3} v_1
\]

Choose $v_1 = 1$:

$$
v^{(1)} = \begin{bmatrix}
1 \\
- \frac{1 - \sqrt{7}}{3}
\end{bmatrix}
$$

Similarly, for $\lambda_2 = 1 - \sqrt{7}$:

$$
A - \lambda_2 I = \begin{bmatrix}
2 - (1 - \sqrt{7}) & 3 \\
2 & - (1 - \sqrt{7})
\end{bmatrix}
= \begin{bmatrix}
1 + \sqrt{7} & 3 \\
2 & -1 + \sqrt{7}
\end{bmatrix}
$$

From the first row:

$$
(1 + \sqrt{7}) v_1 + 3 v_2 = 0 \Rightarrow v_2 = - \frac{1 + \sqrt{7}}{3} v_1
$$

Set $v_1 = 1$:

$$
v^{(2)} = \begin{bmatrix}
1 \\
- \frac{1 + \sqrt{7}}{3}
\end{bmatrix}
$$

Step 3: Write the General Solution

The general solution for a system with distinct real eigenvalues and eigenvectors is:

$$
\mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{v}^{(1)} + c_2 e^{\lambda_2 t} \mathbf{v}^{(2)}
$$

where c_1, c_2 are arbitrary constants determined by initial conditions.

Expressed explicitly:

$$
\boxed{
\begin{aligned}
x(t) &= c_1 e^{(1 + \sqrt{7}) t} \times 1 + c_2 e^{(1 - \sqrt{7}) t} \times 1 \\
y(t) &= c_1 e^{(1 + \sqrt{7}) t} \left(- \frac{1 - \sqrt{7}}{3} \right) + c_2 e^{(1 - \sqrt{7}) t} \left(- \frac{1 + \sqrt{7}}{3} \right)
\end{aligned}
}
$$

This solution describes the entire family of trajectories governed by the initial state of the system.

Part (b): Sketching the Phase Portrait

Understanding the qualitative behavior of solutions is as crucial as deriving explicit formulas. The phase portrait offers a visual snapshot of the system's dynamics, revealing tendencies such as whether solutions tend toward equilibrium points, spiral, or diverge.

Step 1: Identify Equilibrium Point

The only equilibrium point occurs at $\mathbf{X} = \mathbf{0}$, since:

$$
A \mathbf{X} = \mathbf{0} \Rightarrow \mathbf{X} = \mathbf{0}
$$

Step 2: Determine the Nature of the Equilibrium

The eigenvalues are real and distinct:

- $\lambda_1 = 1 + \sqrt{7} \approx 1 + 2.6458 = 3.6458$
- $\lambda_2 = 1 - \sqrt{7} \approx 1 - 2.6458 = -1.6458$

Because one eigenvalue is positive and the other negative, the equilibrium is a saddle point. The trajectories will be attracted along the eigenvector associated with the negative eigenvalue and repelled along the eigenvector associated with the positive eigenvalue.

Step 3: Sketching the Trajectories

Based on the eigenvalues and eigenvectors:

- Stable manifold: along the eigenvector corresponding to λ_2, solutions tend toward the equilibrium as $t \to \infty$.
- Unstable manifold: along the eigenvector corresponding to λ_1, solutions diverge as $t \to \infty$.

Visual features:

- The phase plane exhibits a saddle point at the origin.
- Trajectories approach the origin along

Consider The Planar Linear System X Ax Where 2 3 A 3 2 A Find The General Solution B Sketch

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