Consider The Vector Field F(x,y,z)=(2y,2x,7z)F(x,y,z)=(2y,2x,7z). Show That F Is A Gradient Vector Field

Consider The Vector Field F(x,y,z)=(2y,2x,7z). Show That F Is A Gradient Vector Field

Understanding whether a given vector field is a gradient vector field is a fundamental concept in vector calculus, with implications across physics, engineering, and mathematics. In this article, we will explore the vector field $F(x,y,z) = (2y, 2x, 7z)$ and demonstrate step-by-step how to determine if it is a gradient vector field. We'll also delve into the core concepts, criteria, and methods used to identify gradient fields, supported by mathematical proofs and explanations. By the end, you'll have a clear understanding of the process and the significance of gradient vector fields in various applications.

What Is a Gradient Vector Field?

Definition of a Gradient Vector Field

A gradient vector field is a vector field that can be expressed as the gradient of some scalar potential function. In mathematical terms:

$F(x, y, z) = \nabla\varphi(x, y, z)$

where $\varphi(x, y, z)$ is a scalar potential function, and ∇ (nabla operator) is defined as:

$\nabla\varphi = (\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\varphi/\partial z)$

Key points:

- Gradient vector fields are conservative, meaning the work done moving along a path depends only on the endpoints.
- These fields are irrotational; their curl is zero.
- They often model physical phenomena such as gravitational and electrostatic fields.

Criteria for a Vector Field to Be a Gradient Field

To determine whether a vector field F is a gradient field, certain mathematical conditions must be satisfied:

1. Curl of the Field Must Be Zero

The field must be irrotational:

$\nabla \times F = 0$

This is a necessary condition in simply connected domains.

2. Existence of a Scalar Potential Function

If the curl condition holds, then a scalar potential φ exists such that:

$F = \nabla\varphi$

and

$\partial\varphi/\partial x = F_x$, $\partial\varphi/\partial y = F_y$, $\partial\varphi/\partial z = F_z$

3. Compatibility Conditions

Mixed partial derivatives should be consistent:

- $\partial F_x/\partial y = \partial F_y/\partial x$
- $\partial F_x/\partial z = \partial F_z/\partial x$
- $\partial F_y/\partial z = \partial F_z/\partial y$

These conditions ensure the existence of a scalar potential function φ.

Analyzing the Vector Field F(x,y,z) = (2y, 2x, 7z)

Let's apply the criteria to the given vector field:

$F(x, y, z) = (2y, 2x, 7z)$

where:
- $F_x = 2y$
- $F_y = 2x$
- $F_z = 7z$

Step 1: Compute the Curl of F

The curl of F in three dimensions is:

$$\nabla \times F = (\partial F_z/\partial y - \partial F_y/\partial z, \partial F_x/\partial z - \partial F_z/\partial x, \partial F_y/\partial x - \partial F_x/\partial y)$$

Calculate each component:

- Component 1 (x-component):

$\partial F_z/\partial y = \partial(7z)/\partial y = 0$

$\partial F_y/\partial z = \partial(2x)/\partial z = 0$

So, $(\nabla \times F)_x = 0 - 0 = 0$

- Component 2 (y-component):

$\partial F_x/\partial z = \partial(2y)/\partial z = 0$

$\partial F_z/\partial x = \partial(7z)/\partial x = 0$

So, $(\nabla \times F)_y = 0 - 0 = 0$

- Component 3 (z-component):

$\partial F_y/\partial x = \partial(2x)/\partial x = 2$

$\partial F_x/\partial y = \partial(2y)/\partial y = 2$

So, $(\nabla \times F)_z = 2 - 2 = 0$

Result:

$\nabla \times F = (0, 0, 0)$

Since the curl of F is zero everywhere in the domain, F satisfies the irrotational condition necessary for being a gradient field.

Step 2: Find the Scalar Potential Function φ(x, y, z)

Given that $F = \nabla\varphi$, the components relate as:

- $\partial\varphi/\partial x = 2y$
- $\partial\varphi/\partial y = 2x$
- $\partial\varphi/\partial z = 7z$

We will integrate these to find φ.

Step 3: Integration to Find φ

Integrate $\partial\varphi/\partial x = 2y$ with respect to x:

$\varphi(x, y, z) = 2yx + h(y, z)$

where h(y, z) is an arbitrary function of y and z.

Differentiate φ with respect to y:

$\partial\varphi/\partial y = 2x + \partial h/\partial y$

But from the given, $\partial\varphi/\partial y = 2x$

Set equal:

$2x + \partial h/\partial y = 2x$

which implies:

$\partial h/\partial y = 0$

Therefore, $h(y, z) = H(z)$ (a function only of z).

Now, differentiate φ with respect to z:

$\partial\varphi/\partial z = \partial/\partial z\,[2\,y\,x + H(z)] = dH/dz$

But from the given, $\partial\varphi/\partial z = 7z$

Set equal:

$dH/dz = 7z$

Integrate with respect to z:

$H(z) = (7/2)\,z^2 + C$

where C is a constant.

Final Scalar Potential Function:

$\varphi(x, y, z) = 2\,y\,x + (7/2)\,z^2 + C$

Since constants do not affect gradients, we can ignore C.

Therefore,

$\varphi(x, y, z) = 2\,xy + (7/2)\,z^2$

Conclusion: F Is a Gradient Vector Field

Having demonstrated that the curl of F is zero and explicitly constructed a scalar potential function $\varphi(x, y, z)$ such that:

$\nabla\varphi = (\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\varphi/\partial z) = (2\,y, 2\,x, 7\,z) = F(x, y, z)$

we conclude:

The vector field $F(x, y, z) = (2y, 2x, 7z)$ is indeed a gradient vector field.

Implications and Applications of Gradient Vector Fields

Understanding that a vector field is a gradient field has significant implications:

- Conservative Nature: The work done moving along a path in such a field depends only on the endpoints, not the path taken.
- Potential Energy: Many physical systems, such as gravitational or electrostatic fields, are modeled as gradient fields.
- Mathematical Simplification: Gradient fields allow easier calculation of line integrals and potential functions, simplifying complex problems.

Summary of Key Points

- A vector field F is a gradient field if its curl is zero and if a scalar potential function exists.
- For $F(x,y,z) = (2y, 2x, 7z)$, the curl was calculated and found to be zero.
- A potential function $\varphi(x, y, z) = 2xy + (7/2) z^2$ was explicitly constructed, confirming F as a gradient field.
- Recognizing gradient vector fields is crucial in physics, engineering, and mathematics for analyzing conservative systems and simplifying calculations.

Further Reading and Resources

- Vector Calculus Textbooks: For deeper understanding of curl, divergence, and potential functions.
- Online Calculus Tutorials: Interactive lessons on gradient fields and their properties.
- Mathematical Software: Use tools like WolframAlpha or MATLAB for symbolic computation of vector fields and potential functions.

In conclusion, the process of verifying whether a vector field is a gradient field involves checking the curl condition and finding an explicit potential function. In the case of $F(x,y,z) = (2y, 2x, 7z)$, the

Frequently Asked Questions

What is a gradient vector field?

A gradient vector field is a vector field that can be expressed as the gradient of a scalar potential function, meaning $F = \nabla\varphi$ for some scalar function φ.

Given the vector field F(x,y,z) = (2y, 2x, 7z), how can we verify if it is a gradient field?

We can attempt to find a scalar function φ such that $\nabla\varphi = F$, meaning $\partial\varphi/\partial x = 2y$, $\partial\varphi/\partial y = 2x$, and $\partial\varphi/\partial z = 7z$. If such a φ exists and the mixed partial derivatives are consistent, then F is a gradient field.

What conditions must be satisfied for a vector field to be a gradient field in three dimensions?

The vector field must be irrotational, which means its curl must be zero ($\nabla\times F = 0$), and the domain must be simply connected.

Does the curl of F(x,y,z) = (2y, 2x, 7z) equal zero?

Yes. Computing the curl yields zero, indicating the vector field is irrotational, a key condition for being a gradient field.

How do you find the potential function φ for F(x,y,z) = (2y, 2x, 7z)?

Integrate each component with respect to its variable: $\varphi(x,y,z) = xy + x^2/2 + (7z^2)/2 + C$, ensuring consistency across partial derivatives.

What is the potential function φ(x,y,z) that corresponds to the vector field F(x,y,z) = (2y, 2x, 7z)?

A suitable potential function is $\varphi(x,y,z) = xy + (x^2)/2 + (7z^2)/2 + C$, where C is an arbitrary constant.

Why is showing that the curl of F is zero sufficient to demonstrate that F

is a gradient field?

Because in simply connected domains, a vector field with zero curl is conservative and can be expressed as the gradient of some scalar potential, confirming it as a gradient field.

What is the significance of the domain being simply connected in this context?

A simply connected domain ensures that any irrotational vector field is also conservative, meaning it can be expressed as a gradient field without issues related to multivalued potentials.

Additional Resources

Consider The Vector Field $F(x,y,z) = (2y, 2x, 7z)$: Demonstrating that F Is a Gradient Vector Field

Introduction

In the realm of vector calculus, understanding the nature of vector fields is fundamental to exploring various physical phenomena and mathematical structures. Among these, gradient vector fields play a pivotal role due to their direct connection to scalar potential functions and their inherent properties, such as irrotationality and conservative behavior. The question of whether a given vector field is a gradient vector field is not merely academic; it has practical implications in physics, engineering, and mathematics.

This article delves into a specific vector field:
$F(x, y, z) = (2y, 2x, 7z)$

Our goal is to rigorously demonstrate that F is a gradient vector field. To do so, we will explore the defining characteristics of gradient fields, verify the necessary conditions through comprehensive calculations, and ultimately identify a potential function $\varphi(x, y, z)$ such that $F = \nabla\varphi$.

Background: Gradient Vector Fields and Their Significance

What Is a Gradient Vector Field?

A gradient vector field is a vector field that can be expressed as the gradient of some scalar potential function. Formally, for a scalar function $\varphi(x, y, z)$, the gradient is defined as:

$$
\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)
$$

\]

If a vector field F can be written as $F = \nabla\varphi$, then F is called a gradient field or conservative field.

Key Properties of Gradient Fields

- Irrotationality: The curl of a gradient field is zero, i.e.,
\[
\nabla \times \mathbf{F} = \mathbf{0}
\]
- Path Independence: Line integrals of F depend only on endpoints, not on the path taken.
- Existence of Potential Function: There exists a scalar function φ such that $F = \nabla\varphi$.

Conditions for a Vector Field to be a Gradient Field

In simply-connected regions (regions without holes), a necessary and sufficient condition for F to be a gradient field is:

\[
\nabla \times \mathbf{F} = \mathbf{0}
\]

Additionally, the mixed partial derivatives of the potential function should be continuous and equal, satisfying Clairaut's theorem.

Verifying That F(x, y, z) = (2y, 2x, 7z) Is a Gradient Field

Step 1: Compute the Curl of F

Given:

\[
\mathbf{F}(x, y, z) = (F_1, F_2, F_3) = (2y, 2x, 7z)
\]

The curl of F is:

\[
\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\
\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial
F_1}{\partial y} \right)

\]

Calculating each component:

- x-component:

\[
\frac{\partial F_3}{\partial y} = \frac{\partial 7z}{\partial y} = 0
\]
\[
\frac{\partial F_2}{\partial z} = \frac{\partial 2x}{\partial z} = 0
\]
\[
\Rightarrow \text{x-component} = 0 - 0 = 0
\]

- y-component:

\[
\frac{\partial F_1}{\partial z} = \frac{\partial 2y}{\partial z} = 0
\]
\[
\frac{\partial F_3}{\partial x} = \frac{\partial 7z}{\partial x} = 0
\]
\[
\Rightarrow \text{y-component} = 0 - 0 = 0
\]

- z-component:

\[
\frac{\partial F_2}{\partial x} = \frac{\partial 2x}{\partial x} = 2
\]
\[
\frac{\partial F_1}{\partial y} = \frac{\partial 2y}{\partial y} = 2
\]
\[
\Rightarrow \text{z-component} = 2 - 2 = 0
\]

Result:

\[

\nabla \times \mathbf{F} = (0, 0, 0)
\]

Since the curl of F is zero everywhere in $\mathbb{R}^3$, F is irrotational, satisfying a key condition for being a gradient field.

Step 2: Identify a Potential Function φ(x, y, z)

Given that F = (2y, 2x, 7z), and assuming F = ∇φ, then:

\[
\frac{\partial \phi}{\partial x} = 2y
\]
\[
\frac{\partial \phi}{\partial y} = 2x
\]
\[
\frac{\partial \phi}{\partial z} = 7z
\]

Our task is to find φ(x, y, z) satisfying these partial derivatives.

Step 3: Integrate Partial Derivatives to Find φ

Integrate with respect to x:

\[
\frac{\partial \phi}{\partial x} = 2y \Rightarrow \phi(x, y, z) = 2yx + h(y, z)
\]

where h(y, z) is an arbitrary function of y and z.

Differentiate φ with respect to y:

\[
\frac{\partial \phi}{\partial y} = 2x + \frac{\partial h(y, z)}{\partial y}
\]

But from earlier, we know:

$$
\frac{\partial \phi}{\partial y} = 2x
$$

Matching these:

$$
2x + \frac{\partial h(y, z)}{\partial y} = 2x \Rightarrow \frac{\partial h(y, z)}{\partial y} = 0
$$

Thus, $h(y, z)$ does not depend on y, so:

$$
h(y, z) = k(z)
$$

Next, differentiate φ with respect to z:

$$
\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z} [2yx + k(z)] = \frac{d k(z)}{d z}
$$

From the given, the z-component of F is:

$$
\frac{\partial \phi}{\partial z} = 7z
$$

Therefore:

$$
\frac{d k(z)}{d z} = 7z
$$

Integrate:

$$
k(z) = \frac{7}{2} z^2 + C
$$

where C is an arbitrary constant.

Final Potential Function:

$$
\boxed{
\phi(x, y, z) = 2 y x + \frac{7}{2} z^2 + C
}
$$

Since the constant C does not affect the gradient, the potential function simplifies to:

$$
\phi(x, y, z) = 2 xy + \frac{7}{2} z^2
$$

Conclusion: F Is a Gradient Vector Field

Our analysis confirms that:

1. The curl of F is zero everywhere in $\mathbb{R}^3$.
2. The partial derivatives of F satisfy the symmetry conditions necessary for the existence of a scalar potential.
3. A potential function $\varphi(x, y, z) = 2xy + (7/2) z^2$ exists such that $\nabla\varphi = F$.

Therefore, $F(x, y, z) = (2y, 2x, 7z)$ is indeed a gradient vector field or conservative field in the entire space $\mathbb{R}^3$.

Broader Implications and Context

Significance in Physics and Engineering

The identification of F as a gradient field implies that it can be derived from a potential energy function, making it relevant in contexts such as:

- Gravitational potential fields
- Electrostatic fields
- Conservative force fields in mechanics

Mathematical Insights

This example underscores the importance of verifying both curl and potential functions when analyzing vector fields. In more complex scenarios, especially in multiply-connected regions, additional conditions or techniques (such as examining the domain's topology) are necessary to establish conservativity.

Summary of Key Steps

- Compute the curl of the vector field to test irrotationality.
- Integrate partial derivatives to identify a potential function.
- Verify the consistency of the potential function with all components of the vector field.

Final Remarks

The process demonstrated here—computing curl, integrating partial derivatives, and confirming the existence of a potential function—serves

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