Determine Whether Or Not The Vector Field Is Conservative. If It Is Conservative, Find A Function F Such

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In vector calculus, understanding whether a given vector field is conservative is a fundamental step towards analyzing the properties of the field, such as potential functions, line integrals, and applications in physics, especially in fields like electromagnetism and fluid dynamics. A conservative vector field is one where the line integral between two points is path-independent, meaning the work done by the field in moving an object between any two points depends only on those points and not on the path taken. This property is closely related to the existence of a scalar potential function F, which simplifies the analysis of the field. In this article, we explore the criteria for determining whether a vector field is conservative and, if it is, how to find the potential function F.

Understanding Conservative Vector Fields

Definition of a Conservative Vector Field

A vector field $\vec{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is called conservative if there exists a scalar function $F(x, y, z)$ such that:

- $\vec{F} = \nabla F$, meaning $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$, and $R = \frac{\partial F}{\partial z}$.
- The line integral of $\vec{F}$ over any curve C from point A to point B depends only on A and B, not on the path taken.

Physical and Mathematical Significance

Conservative fields often represent physical phenomena where energy is conserved, such as gravitational and electrostatic fields. Mathematically, they simplify calculations of work, flux, and potential differences, as the existence of a potential function F enables easy computation of line integrals through evaluation at endpoints.

Criteria for a Vector Field to Be Conservative

Necessary Conditions in $\mathbb{R}^3$

For a smooth vector field $\vec{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the following are necessary conditions for $\vec{F}$ to be conservative:

1. $\vec{F}$ must be defined on a simply connected domain. This means the domain has no "holes" or obstacles that prevent continuous paths from being contracted to a point.

2. The curl of $\vec{F}$ must be zero:

$$
\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} = \vec{0}
$$

If these conditions are met, the field is conservative within that domain. Conversely, in simply connected domains, a curl-free vector field is indeed conservative.

In $\mathbb{R}^2$

For a vector field $\vec{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, the condition reduces to:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0
$$

and the domain must be simply connected for the field to be conservative.

Procedure to Determine if a Vector Field Is Conservative

Step 1: Verify the Domain

- Ensure the domain of $\vec{F}$ is simply connected (e.g., $\mathbb{R}^3$, $\mathbb{R}^2$ minus a point or a line).
- If the domain is not simply connected, the field may not be conservative, even if curl is zero.

Step 2: Calculate the Curl

Compute $\nabla \times \vec{F}$. If the curl is not zero, the field is not conservative.

Step 3: Confirm Zero Curl and Domain Connectivity

- If $\nabla \times \vec{F} = \vec{0}$ and the domain is simply connected, then $\vec{F}$ is conservative.
- If not, the field is non-conservative.

Finding the Potential Function F if $\vec{F}$ Is Conservative

Step 1: Integrate P with respect to x

Start by integrating the x-component $P(x, y, z)$ with respect to x:

$$
F(x, y, z) = \int P(x, y, z) \, dx + g(y, z)
$$

Here, $g(y, z)$ is an arbitrary function of y and z, arising because the partial derivative with respect to x eliminates any terms that depend solely on y and z.

Step 2: Use Q to Find $g(y, z)$

Differentiate the obtained F with respect to y:

$$
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left(\int P \, dx + g(y, z) \right) = \int \frac{\partial P}{\partial y} dx + \frac{\partial g}{\partial y}
$$

Compare this with $Q(x, y, z)$. Since:

$$
Q = \frac{\partial F}{\partial y}
$$

we can solve for $g(y, z)$ by integrating and matching terms.

Step 3: Use R to Find $h(z)$

Similarly, differentiate F with respect to z and compare with $R(x, y, z)$. This helps determine any remaining functions of z, completing the potential function F.

Worked Example

Given Vector Field

Suppose $\vec{F} = (2xy)\mathbf{i} + (x^2 + z)\mathbf{j} + (y + xz)\mathbf{k}$.

Step 1: Check Curl

$$
\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
$$

Calculate:

$$
\frac{\partial R}{\partial y} = \frac{\partial (y + xz)}{\partial y} = 1

$$
\frac{\partial Q}{\partial z} = \frac{\partial (x^2 + z)}{\partial z} = 1
$$

$$
\frac{\partial P}{\partial z} = \frac{\partial (2xy)}{\partial z} = 0
$$

$$
\frac{\partial R}{\partial x} = \frac{\partial (y + xz)}{\partial x} = z
$$

$$
\frac{\partial Q}{\partial x} = \frac{\partial (x^2 + z)}{\partial x} = 2x
$$

$$
\frac{\partial P}{\partial y} = \frac{\partial (2xy)}{\partial y} = 2x
$$

So,

$$
\nabla \times \vec{F} = (1 - 1)\mathbf{i} + (0 - z)\mathbf{j} + (2x - 2x)\mathbf{k} = 0\mathbf{i} - z\mathbf{j} + 0\mathbf{k}
$$

Since the curl is not zero unless $z=0$, the field is not conservative in $\mathbb{R}^3$ generally. However, if we're restricted to the

Frequently Asked Questions

How can I determine if a vector field is conservative?

To determine if a vector field is conservative, check if its curl is zero (in 3D) or if the line integral around any closed loop is zero. In simply connected regions, a zero curl implies the field is conservative.

What conditions must a vector field satisfy to be considered conservative?

A vector field is conservative if it can be expressed as the gradient of a scalar potential function, which typically requires the curl of the field to be zero and the domain to be simply connected.

If a vector field F is conservative, how do I find the potential function F?

To find the potential function F, integrate the components of the vector field with respect to their variables, ensuring consistency across partial derivatives to confirm the function is well-defined.

Can a vector field with zero curl be non-conservative?

In simply connected regions, a vector field with zero curl is conservative. However, in multiply connected regions, zero curl does not necessarily guarantee the field is conservative.

What is the significance of a conservative vector field in physics?

Conservative vector fields are important because they imply the existence of a potential energy function, making work path-independent and simplifying analyses in fields like gravity and electromagnetism.

How do I verify if the line integral of a vector field over a closed loop is zero?

Calculate the line integral over the closed loop. If the integral equals zero, and the domain is simply connected, it indicates the field may be conservative.

What are common examples of conservative vector fields?

Examples include gravitational fields, electrostatic fields, and any gradient fields derived from scalar potential functions like $F = \nabla\varphi$.

Additional Resources

Determine Whether Or Not The Vector Field Is Conservative. If It Is Conservative, Find A Function F Such

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Introduction

In vector calculus, the concept of a conservative vector field is fundamental due to its profound implications in physics, engineering, and mathematics. Understanding whether a vector field is conservative affects how we approach problems involving work, potential functions, and flux. This comprehensive review aims to elucidate the criteria for identifying conservative vector fields, explore their properties, and demonstrate methods to find an associated potential function \(F\) when the field is conservative.

What Is a Conservative Vector Field?

A vector field \(\mathbf{F}(\mathbf{r})\) in \(\mathbb{R}^n\) assigns a vector to each point in space. A vector field is called conservative if it can be expressed as the gradient of some scalar potential function \(F(\mathbf{r})\):

\[

$$
\mathbf{F} = \nabla F
$$

This implies that the vector field is irrotational (has zero curl in three dimensions) and the line integral of the field between two points depends only on the endpoints, not the path taken.

Key Point:
Conservative vector fields are exact fields; their line integrals are path-independent.

Physical and Mathematical Significance

- In physics, conservative fields correspond to force fields where the work done in moving an object between two points is independent of the path taken, such as gravitational or electrostatic fields.
- Mathematically, the existence of a scalar potential simplifies analysis, enabling us to evaluate integrals easily and analyze the behavior of the field.

Criteria for a Vector Field to Be Conservative

Determining whether a vector field is conservative involves verifying certain properties. The criteria depend on the domain and the nature of the vector field.

1. Continuity and Domain Considerations

- The domain Ω where the vector field is defined plays a critical role.
- For simply connected domains (domains without holes), the criteria are straightforward.
- For multiply connected domains, additional considerations are necessary.

2. Necessary and Sufficient Conditions

In $\mathbb{R}^3$, for a continuously differentiable vector field $\mathbf{F} = (P, Q, R)$, the following are key:

- Curl Condition:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

- Connected Domain:
Ω must be simply connected (no holes).

In two dimensions, for $\mathbf{F} = (P, Q)$:

- Curl (or Rotation):

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0
$$

- Simply Connected Domain:
The region must be simply connected for the field to be conservative.

Step-by-Step Process to Determine if a Vector Field is Conservative

Step 1: Check Continuity and Differentiability

- Verify the vector field $\mathbf{F}$ has continuous partial derivatives in the domain of interest.

Step 2: Compute Curl

- Calculate the curl $\nabla \times \mathbf{F}$:
- In $\mathbb{R}^3$,
$$
\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
$$
- In $\mathbb{R}^2$, check if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

- If the curl is not zero at any point, then $\mathbf{F}$ is not conservative.

Step 3: Verify Domain Connectivity

- Confirm the domain Ω is simply connected.
- If the domain has holes or is not simply connected, a zero curl does not necessarily imply the field is conservative.

Step 4: Path Independence of Line Integral

- Alternatively, verify whether the line integral of $\mathbf{F}$ around closed loops in Ω is zero.
- If the integral around every closed loop is zero, $\mathbf{F}$ is conservative.

Finding the Potential Function F

If $\mathbf{F}$ is confirmed to be conservative, the next step is to find a scalar function F such that:

$$
\nabla F = \mathbf{F}
$$
or equivalently,

$$
\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q, \quad \text{(and } \frac{\partial F}{\partial z} = R \text{ in 3D).}
$$

General Procedure:

1. Integrate one component:
- For example, in $\mathbb{R}^2$, integrate P with respect to x:
$$
F(x, y) = \int P(x, y) \, dx + C(y)
$$
- $C(y)$ is an arbitrary function of y, arising because the partial derivative with respect to x might eliminate some y-dependence.

2. Differentiate the obtained F:
- Calculate $\frac{\partial F}{\partial y}$:
$$
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left(\int P(x, y) \, dx + C(y) \right)
$$
- Set this equal to $Q(x, y)$ and solve for $C(y)$:
$$
Q(x, y) = \frac{\partial}{\partial y} \left(\int P(x, y) \, dx \right) + C'(y)
$$
- Integrate to find $C(y)$.

3. Repeat for other components:
- In three dimensions, similarly integrate P, then use Q and R to determine the potential function's form.

Examples Illustrating the Process

Example 1: Simple 2D Field

Suppose:

$$
\mathbf{F}(x, y) = (2xy, x^2)
$$

Step 1: Check curl:

$$
\frac{\partial Q}{\partial x} = 2x,\quad \frac{\partial P}{\partial y} = 2x
$$

Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, the curl is zero.

Step 2: Confirm domain is simply connected (assume $\mathbb{R}^2$).

Step 3: Find F:

$$
F(x, y) = \int P(x, y) \, dx = \int 2xy \, dx = x^2 y + C(y)
$$

Differentiate with respect to y:

$$
\frac{\partial F}{\partial y} = x^2 + C'(y)
$$

Set equal to $Q(x, y) = x^2$:

$$
x^2 + C'(y) = x^2 \quad \Rightarrow \quad C'(y) = 0
$$

Integrate:

$$
C(y) = \text{constant}
$$

Result:

$$
\boxed{
F(x, y) = x^2 y + \text{constant}
}
$$

Example 2: 3D Field

Given:

$$
\mathbf{F}(x, y, z) = (yz, xz, xy)
$$

Step 1: Compute curl:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
$$

Calculate each component:

- $\frac{\partial R}{\partial y} = \frac{\partial xy}{\partial y} = x$
- $\frac{\partial Q}{\partial z} = \frac{\partial xz}{\partial z} = x$

First component:

\[
x - x = 0
\]

Similarly,

- \(\frac{\partial P}{\partial z} = \frac{\partial yz}{\partial z} = y\)
- \(\frac{\partial R}{\partial x} = \frac{\partial xy}{\partial x} = y\)

Second component:

\[
y - y = 0
\]

- \(\frac{\partial Q}{\partial x} = z\)
- \(\frac{\partial P}{\partial y} = z\)

Third component

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