Evaluate 2 0 X 0 X2 Y2 Dy Dx 2 2 4x2 0 X2 Y2 Dy Dx.Evaluate The Iterated Integral By Converting To Polar

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When encountering complex double integrals, especially those involving quadratic expressions like $X^2 + Y^2$, converting to polar coordinates often simplifies the evaluation process significantly. This approach leverages the symmetry inherent in many regions bounded by circles or parts of circles and transforms complicated algebraic expressions into more manageable forms. In this comprehensive guide, we will explore how to evaluate the given double integral by converting it into polar coordinates, emphasizing step-by-step procedures, key concepts, and practical tips to enhance your understanding of multivariable calculus.

Understanding the Integral and Its Region

Before diving into the conversion process, it's crucial to understand the integral's structure and the region over which it is evaluated.

The Integral Expression

The integral presented is:

$$
\int_{x=0}^{2} \int_{y=0}^{x^2} (X^2 + Y^2) \, dy \, dx
$$

or, as per the initial notation:

$$
\int_{x=0}^{2} \int_{y=0}^{x^2} (X^2 + Y^2) \, dy \, dx
$$

This is an iterated integral with the region defined by the bounds:

- x varies from 0 to 2.
- For each fixed x, y varies from 0 to x^2.

The region R can thus be described as:

$$
R = \{ (x,y) \mid 0 \leq x \leq 2, \quad 0 \leq y \leq x^2 \}
$$

Graphically, this region is bounded between the parabola $y = x^2$ and the x-axis, from $x=0$ to $x=2$.

Why Convert to Polar Coordinates?

Converting to polar coordinates is especially advantageous when:

- The region is circular or bounded by curves involving $x^2 + y^2$.
- The integrand involves $X^2 + Y^2$, which simplifies to r^2 in polar coordinates.
- The boundary curves are circles, sectors, or parts thereof.

In this integral, the integrand contains $X^2 + Y^2$, and the boundary $y = x^2$ resembles a parabola, which is not circular—so direct conversion requires careful attention. However, parts of the region can be better understood or approximated via polar coordinates, especially if the region can be expressed in terms of r and θ.

Step-by-Step Guide to Converting and Evaluating the Integral

1. Express the Region in Polar Coordinates

Recall the transformation formulas:

$$
x = r \cos \theta,\quad y = r \sin \theta
$$

and the Jacobian for the change of variables:

$$
dA = r\, dr\, d\theta
$$

Our goal is to describe the region R in (r, θ) coordinates.

2. Analyze the Boundary Curves

- The boundary $y=0$:

$$
r \sin \theta = 0 \implies \sin \theta = 0 \implies \theta = 0,\pi, 2\pi,\dots
$$

In our region, since $x \geq 0$ and $y \geq 0$, the relevant θ values are:

$$
0 \leq \theta \leq \frac{\pi}{2}
$$

- The boundary $y = x^2$:

Expressed in polar:

$$
r \sin \theta = (r \cos \theta)^2 = r^2 \cos^2 \theta
$$
$$
r \sin \theta = r^2 \cos^2 \theta
$$

Dividing both sides by r (assuming $r \geq 0$):

$$
\sin \theta = r \cos^2 \theta
$$
$$
r = \frac{\sin \theta}{\cos^2 \theta}
$$

Note: For $\theta \in [0, \pi/2]$, $\cos \theta > 0$, so the expression is valid.

This boundary describes the maximum radial extent r for each θ:

$$
r \leq \frac{\sin \theta}{\cos^2 \theta}
$$

- The boundary $x=2$:

$$
x=2 \implies r \cos \theta=2 \implies r= \frac{2}{\cos \theta}
$$

Thus, for each θ, the region is bounded by:

$$
r \leq \frac{\sin \theta}{\cos^2 \theta} \quad \text{(parabola boundary)}
$$
and
$$
r \leq \frac{2}{\cos \theta} \quad \text{(vertical boundary at } x=2)
$$

However, since the region is between $x=0$ and $x=2$, and $y \ge 0$, the radial bounds are:

$$
0 \leq r \leq \min \left(\frac{\sin \theta}{\cos^2 \theta}, \frac{2}{\cos \theta} \right)
$$

for $\theta \in [0, \pi/2]$.

3. Determine the Angular Range

- Since $x \geq 0$ and $y \geq 0$, the relevant θ is from 0 to $\pi/2$.

- At $\theta = 0$:

$$
r \leq \frac{\sin 0}{\cos^2 0} = 0
$$

- At $\theta = \pi/2$:

$$
r \leq \frac{\sin (\pi/2)}{\cos^2 (\pi/2)} = \frac{1}{0} \to \infty
$$

but the boundary at $x=2$:

$$
r \leq \frac{2}{\cos (\pi/2)} \to \infty
$$

indicating the region extends towards $\theta \to \pi/2$.

For practical purposes, the region corresponds to θ in $[0, \pi/2]$, with the upper bounds for r as described.

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4. Set Up the Integral in Polar Coordinates

The original integrand:

$$
X^2 + Y^2 = r^2
$$

The differential area element:

$$
dy\, dx = r\, dr\, d\theta
$$

The double integral becomes:

$$
\int_{\theta=0}^{\pi/2} \int_{r=0}^{r_{\max}(\theta)} r^2 \times r\, dr\, d\theta = \int_{0}^{\pi/2} \int_{0}^{r_{\text{max}}(\theta)} r^3\, dr\, d\theta
$$

where:

$$
r_{\text{max}}(\theta) = \min \left(\frac{\sin \theta}{\cos^2 \theta}, \frac{2}{\cos \theta} \right)
$$

---

5. Evaluate the Inner Integral

The inner integral:

$$
\int_{0}^{r_{\text{max}}(\theta)} r^3\, dr = \left[\frac{r^4}{4} \right]_0^{r_{\text{max}}(\theta)} = \frac{ (r_{\text{max}}(\theta))^4 }{4}
$$

Thus, the entire integral simplifies to:

$$
\frac{1}{4} \int_{0}^{\pi/2} \left(r_{\text{max}}(\theta) \right)^4 d\theta
$$

---

6. Express \(r_{\text{max}}(\theta)\) and Final Integration

The maximum radius is the minimum of the two boundary functions:

\[
r_1(\theta) = \frac{\sin \theta}{\cos^2 \theta}
\]
\[
r_2(\theta) = \frac{2}{\cos \theta}
\]

Identify the \(\theta\) where these two are equal:

\[
\frac{\sin \theta}{\cos^2 \theta} = \frac{2}{\cos \theta}
\]

Multiply both sides by \(\cos^2 \theta\):

\[
\sin \theta = 2 \cos \theta
\]

Dividing both sides by \(\cos \theta\) (where \(\cos \theta \ne 0\)):

\[
\tan \theta = 2
\]

So,

\[
\theta_c = \

Frequently Asked Questions

What is the primary goal when evaluating the integral 2 0 X 0 X2 Y2 Dy Dx by converting to polar coordinates?

The primary goal is to simplify the double integral by changing from Cartesian coordinates to polar coordinates, which often makes the limits and integrand easier to evaluate, especially when dealing with circular or radial symmetry.

How do you convert the Cartesian limits X=0 to X=2 and Y=2 to Y=Y in the integral into polar coordinates?

You express X and Y in terms of r and θ (X= r cos θ, Y= r sin θ). The limits are then transformed based on the region's boundaries, often involving inequalities like r from 0 to a certain radius and θ over a specific angle range, depending on the region.

What is the general process for converting a Cartesian double integral to a polar integral?

The process involves replacing X and Y with r cos θ and r sin θ, respectively, computing the Jacobian determinant (which is r), and transforming the limits accordingly to describe the same region in polar coordinates.

Why is the Jacobian determinant equal to r when converting to polar coordinates?

Because in polar coordinates, the area element dA = dx dy transforms to r dr dθ due to the Jacobian determinant of the transformation, accounting for the change in area element from Cartesian to polar coordinates.

When evaluating the integral by converting to polar coordinates, how do you determine the new limits of integration?

You analyze the region in the xy-plane, express its boundaries in terms of r and θ, and then set the limits for r and θ to cover exactly the same region, often starting from r=0 to the outer boundary and θ over the angular span of the region.

What are common challenges faced when converting and evaluating integrals in polar coordinates?

Challenges include accurately determining the region's boundaries in polar form, correctly setting the limits for r and θ, and managing integrand complexities that depend on r and θ, especially for irregular regions.

Can you provide a brief overview of the steps to evaluate an integral over a circular region by converting to polar coordinates?

Yes, first identify the region and express its boundaries in terms of r and θ. Next, convert the integrand to polar form. Then, set the integration limits for r and θ based on the region. Finally, include the Jacobian r and evaluate the double integral using these polar limits.

In the context of the integral provided, what specific region does the conversion to polar coordinates help to describe?

The conversion helps to describe the region bounded by the given limits in the xy-plane, likely a circular or sector-shaped region, making the integral easier to evaluate by capturing the symmetry and simplifying the bounds.

Additional Resources

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Introduction

When dealing with complex double integrals in multivariable calculus, the process of conversion from Cartesian coordinates \((x, y)\) to polar coordinates \((r, \theta)\) often simplifies the integration process significantly. The given expression hints at an iterated integral involving variables \(x\) and \(y\), and its evaluation via polar transformation is a classic method to handle regions with circular symmetry or when the bounds are more naturally expressed in terms of \(r\) and \(\theta\).

This detailed review explores the concepts, techniques, and step-by-step methodology for converting and evaluating an iterated integral via polar coordinates, emphasizing the importance and utility of this approach in solving complex integrals.

Understanding the Integral and Its Components

Before delving into the conversion process, it's crucial to comprehend the structure of the integral and what the notation suggests.

Interpreting the Given Expression

The provided expression appears to be a collection of variables and operators, possibly representing a double integral:

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Evaluate 2 0 X 0 X2 Y2 Dy Dx 2 2 4x2 0 X2 Y2 Dy Dx
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While the notation isn't entirely conventional, we can interpret it as an integral with specific bounds and functions:

- The limits for \(x\) and \(y\) are likely from 0 to some bounds involving \(X\) and \(Y\).
- The integrand involves terms like \(4x^2\), hinting at polynomial functions.
- The repeated mention of \(X2\) and \(Y2\) suggests bounds or functions involving \(X^2\) and \(Y^2\).

Given common practices in multivariable calculus, a typical integral might look like:

$$
\iint_{D} f(x, y) \, dy\, dx
$$
where D is the domain, and the integrand $f(x, y)$ could be a polynomial or a function involving x^2, y^2, etc.

The Rationale for Converting to Polar Coordinates

Converting to polar coordinates is advantageous when:

- The region D is circular or sector-shaped.
- The integrand involves $x^2 + y^2$ or similar expressions.
- The bounds are more naturally expressed in terms of r and θ.

Benefits include:

- Simplification of the integrand, especially when involving squared terms.
- Easier expression of circular regions resulting in more straightforward bounds.
- Reduction of complex algebraic bounds to simple inequalities on r and θ.

Step-by-Step Methodology

1. Expressing Cartesian Coordinates in Polar Coordinates

The fundamental relationships are:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$
and the differential area element:

$$
dx\,dy = r\, dr\, d\theta
$$

This transformation adjusts the integral to:

$$
\iint_{D} f(x, y) \, dx\,dy = \int_{\theta_{a}}^{\theta_{b}}\int_{r_{a}(\theta)}^{r_{b}(\theta)} f(r \cos \theta, r \sin \theta) \, r\, dr\, d\theta
$$

where the bounds $r_{a}(\theta)$, $r_{b}(\theta)$, θ_a, and θ_b are determined by the region D.

2. Identifying the Region of Integration

To convert the integral, the most critical step is understanding the region D.

- If the area is a circle of radius R:

$$
0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi
$$

- If the region is bounded by lines and circles:

The bounds in terms of x and y translate into inequalities involving r and θ. For example, a circle centered at the origin: $x^2 + y^2 \leq R^2$.

- In the case of more complex regions:

Carefully analyze the inequalities to determine how r varies with θ.

3. Transforming the Integrand

Replace x and y in the integrand with their polar equivalents:

$$
f(x, y) \to f(r \cos \theta, r \sin \theta)
$$

For example, if the integrand involves $x^2 + y^2$:

$$
x^2 + y^2 = r^2
$$

which simplifies the integrand substantially.

Applying to the Given Integral

Suppose the integral is over a circular region centered at the origin with radius a. The integral could resemble:

$$
\int_{0}^{2\pi} \int_{0}^{a} f(r, \theta) \, r \, dr\, d\theta
$$

where $f(r, \theta)$ is the transformed integrand.

If the original bounds are more complicated, like a sector or an annular region, similar logic applies but with adjusted bounds.

Example: Evaluating a Specific Integral

Let's illustrate with a concrete example inspired by the hints:

$$
I = \int_{x=0}^{X} \int_{y=0}^{Y} (4x^2 + y^2) \, dy\, dx
$$

Step 1: Recognize the structure—bounds are from 0 to X and 0 to Y, which could form a rectangle or sector.

Step 2: If the bounds involve $x^2 + y^2$ or the region is circular, converting to polar is advantageous.

Step 3: Convert bounds. For simplicity, assume the region is a quarter circle of radius R:

$$
0 \leq r \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}
$$

Step 4: Write the integral in polar coordinates:

$$
I = \int_{0}^{\pi/2} \int_{0}^{R} \left(4(r \cos \theta)^2 + (r \sin \theta)^2\right) r \, dr\, d\theta
$$

which simplifies to:

$$
I = \int_{0}^{\pi/2} \int_{0}^{R} \left(4 r^2 \cos^2 \theta + r^2 \sin^2 \theta\right) r \, dr\, d\theta
$$

$$
= \int_{0}^{\pi/2} \int_{0}^{R} r \left(4 r^2 \cos^2 \theta + r^2 \sin^2 \theta\right) dr\, d\theta
$$

$$
= \int_{0}^{\pi/2} \int_{0}^{R} \left(4 r^3 \cos^2 \theta + r^3 \sin^2 \theta\right) dr\, d\theta
$$

Step 5: Integrate with respect to r:

$$
\int_{0}^{R} 4 r^3 \cos^2 \theta \, dr = 4 \cos^2 \theta \times \frac{R^4}{4} = R^4 \cos^2 \theta
$$

$$
\int_{0}^{R} r^3 \sin^2 \theta \, dr = \sin^2 \theta \times \frac{R^4}{4}
$$

So,

$$
I = \int_{0}^{\pi/2} \left(R^4 \cos^2 \theta + \frac{R^4}{4} \sin^2 \theta \right) d\theta
$$

$$
= R^4 \int_{0}^{\pi/2} \left(\cos^2 \theta + \frac{1}{4} \sin^2 \theta \right) d\theta
$$

Now, evaluate the θ-integrals:

$$
\int_{0}^{\pi/2} \cos^2 \theta \, d\theta = \frac{\pi}{4}
$$

$$
\int_{0}^{\pi/2} \sin^2 \theta \, d\theta = \frac{\pi}{4}
$$

Hence,

$$
I = R^4 \left(\frac{\pi}{4} + \frac{1}{4} \times \frac{\pi}{4} \right) = R^4 \left(\frac{\pi}{4} + \frac{\pi}{16} \right) = R^4 \times \frac{4\pi + \pi}{16} = R^4 \times \frac{5\pi}{16}
$$

Final result:

\[
\boxed{
I = \frac{5 \pi R^4}{16

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