Evaluate Triple Integral Xyzdv Where T Is The Tetrahedron With Vertices (0,0,0) (1,0,0) (1,1,0), (1,0,1)

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When delving into the fascinating world of multivariable calculus, one of the most fundamental yet challenging concepts is evaluating triple integrals over various regions in three-dimensional space. In this article, we will explore how to evaluate the triple integral of the function $xyzdv$ over a specific tetrahedral region T with vertices at (0,0,0), (1,0,0), (1,1,0), and (1,0,1). Understanding this process not only reinforces key integration techniques but also enhances spatial visualization skills, vital for applications in physics, engineering, and mathematics.

Understanding the Region of Integration T

Before jumping into the integral's evaluation, it is crucial to analyze and understand the region T over which we are integrating.

Vertices and Geometric Description of the Tetrahedron

The tetrahedron T is defined by four vertices:

- $V_1 = (0, 0, 0)$
- $V_2 = (1, 0, 0)$
- $V_3 = (1, 1, 0)$
- $V_4 = (1, 0, 1)$

This three-dimensional shape lies within the first octant, bounded by the coordinate planes and the planes passing through these vertices.

Visualizing the Tetrahedron

Imagine the base of the tetrahedron lying on the xy-plane, with vertices at (0,0,0), (1,0,0), and (1,1,0). The fourth vertex at (1,0,1) extends upward along the z-axis from the point (1,0,0). The shape is bounded by:

- The coordinate planes $x=0$, $y=0$, and $z=0$.
- The planes passing through the points $(1,0,0)$, $(1,1,0)$, and $(1,0,1)$.

This region is a tetrahedron with a slanted face connecting the points in three-dimensional space.

Setting Up the Triple Integral

The goal is to evaluate:

$$
\iiint_T xyzdv
$$

where dv represents the volume element in three-dimensional space.

Choosing an Order of Integration

The order of integration significantly affects the ease of calculation. For this region, a logical choice is to integrate with respect to z, then y, and finally x, because the bounds are easier to describe in this order.

Alternatively, other orders can be considered, but this approach simplifies the bounds.

Determining the Limits of Integration

To find the limits:

- z-limits: For fixed x and y, z varies from the bottom plane $z=0$ to the plane passing through $(1,0,1)$.

- Plane passing through $(1,0,1)$: Let's find the equation of the plane containing $(1,0,1)$ and connecting to the base.

The plane through $(1,0,1)$, $(1,0,0)$, and $(1,1,0)$:

- Points:

- $A = (1,0,0)$

- $B = (1,1,0)$

- $C = (1,0,1)$

- Vectors:

- $\vec{AB} = (0,1,0)$

- $\vec{AC} = (0,0,1)$

- Normal vector:

$$
\vec{n} = \vec{AB} \times \vec{AC} = (1,0,0)
$$

Since the normal vector is along the x-direction, the plane equation is:

$$
x = 1
$$

Thus, the plane $x=1$ cuts through the shape at $x=1$.

Now, for x between 0 and 1, the region is bounded by:

- $z=0$ (the base)

- The plane passing through $(1,0,1)$, which is $x=1$, and the other vertices.

But to get the bounds more systematically, let's analyze the faces:

- For fixed x, y varies between 0 and 1, but constrained by the shape.

- The face connecting $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ gives the base $z=0$.

- The face connecting $(1,0,1)$ and the other points defines the upper bounds in z.

Given this, the key is to find the plane passing through $(0,0,0)$, $(1,0,0)$, and $(1,0,1)$, which slices the shape in the yz-plane.

The plane through $(1,0,1)$, $(1,1,0)$, and $(0,0,0)$ can be parametrized, but perhaps a more straightforward approach is to express the bounds in terms of inequalities.

Simplified description of bounds:

- x varies from 0 to 1.

- For each fixed x, y varies from 0 to y_{max}.

- For each fixed x and y, z varies from 0 to the plane passing through $(1,0,1)$, which can be expressed as:

$$
z \leq 1 - (x - 1) \quad \text{(since at $x=1$, $z \leq 1$)}
$$

But, to avoid confusion, perhaps the most precise approach is to find the equations of the bounding planes.

Deriving the Boundary Planes

Let's identify the planes forming the tetrahedron:

1. Base plane: $z=0$

2. Plane through points $(1,0,1), (1,1,0), (0,0,0)$

- Find the equation:

- Vectors:

$$
\vec{P}_1 = (1,0,1) - (0,0,0) = (1,0,1)
$$
$$
\vec{P}_2 = (1,1,0) - (0,0,0) = (1,1,0)
$$

- Normal vector:

$$
\vec{n} = \vec{P}_1 \times \vec{P}_2 =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{vmatrix}
= (0 \times 0 - 1 \times 1)\mathbf{i} - (1 \times 0 - 1 \times 1)\mathbf{j} + (1 \times 1 - 0 \times 1)\mathbf{k}
= (-1)\mathbf{i} - (-1)\mathbf{j} + (1)\mathbf{k} = (-1, 1, 1)
$$

- Equation of the plane:

$$
-1 (x - 0) + 1(y - 0) + 1(z - 0) = 0
$$
$$
-x + y + z = 0
$$
$$
z = x - y
$$

3. Plane through $(1,0,1), (1,1,0), (0,0,0)$:

- Already found as $z = x - y$.

4. Plane through $(1,0,1), (1,1,0), (0,0,0)$:

- Same as above; this represents the boundary between the interior and the exterior of the tetrahedron.

Final bounds:

- x varies from 0 to 1.

- For each fixed x, y varies from 0 to 1 (since at $x=1$, y can go up to 1).

- For each fixed x and y, z varies from 0 up to the plane:

$$z = x - y$$

But since $z \geq 0$, the upper limit is:

$$z_{max} = x - y$$

with the condition that $z_{max} \geq 0$

Frequently Asked Questions

What is the geometric shape of the region T over which the triple integral is evaluated?

T is a tetrahedron with vertices at (0,0,0), (1,0,0), (1,1,0), and (1,0,1).

How can we describe the tetrahedron T in terms of inequalities for x, y, and z?

The tetrahedron T can be described by $0 \leq x \leq 1$, $0 \leq y \leq x$, and $0 \leq z \leq 1 - x$.

What is the order of integration for evaluating the triple

integral over T?

A common order is dz, dy, then dx, integrating z from 0 to 1 - x, y from 0 to x, and x from 0 to 1.

How do you set up the limits of integration for the triple integral $\iiint_T xyzdv$?

The limits are: x from 0 to 1, y from 0 to x, and z from 0 to 1 - x.

What is the value of the triple integral $\iiint_T xyzdv$ over the tetrahedron?

The value is 1/24, obtained by integrating the function xyzd over the specified bounds.

Can you explain the step-by-step process to evaluate the integral $\iiint_T xyzdv$?

Yes, first integrate with respect to z: $\int_0^{1 - x} xy\, z\, dz$, then y: $\int_0^{x}$ (result) dy, and finally x: $\int_0^{1}$ (result) dx, simplifying at each step.

Why is it advantageous to change the order of integration in evaluating this triple integral?

Changing the order can simplify the limits and the integrand, making the integral easier to evaluate or compute.

Are there alternative coordinate systems useful for evaluating the integral over T?

Yes, using barycentric or affine coordinates within the tetrahedron can sometimes simplify the integration process.

What is the significance of understanding the geometry of T when evaluating the triple integral?

Understanding the geometry helps in correctly setting up the limits, choosing the order of integration, and visualizing the volume for accurate computation.

Additional Resources

Evaluate Triple Integral Xyzdv Where T Is The Tetrahedron With Vertices (0,0,0) (1,0,0) (1,1,0), (1,0,1)

When tackling multivariable calculus problems, especially those involving volume integrals, understanding the geometric context is essential. One such problem involves evaluating the triple integral of the function $xyzdv$ over a specific tetrahedron T. The tetrahedron, a polyhedron with four triangular faces, is defined by four vertices: (0,0,0), (1,0,0), (1,1,0), and (1,0,1). This problem combines geometric intuition with algebraic manipulation, illustrating the power of calculus to quantify complex three-dimensional regions.

In this article, we will explore step-by-step how to evaluate the integral $\iiint_T xyzdv$, where T is the tetrahedron with the given vertices. We will interpret the geometric shape, set up the limits of integration through appropriate coordinate transformations, and execute the integral calculation meticulously. This process illuminates the interplay between geometric shapes and calculus, providing valuable insights into multidimensional integration.

Understanding the Geometric Region T

Before diving into the integrals, it’s vital to comprehend the shape and boundaries of the region T. The vertices provided are:

- $V_0 = (0, 0, 0)$
- $V_1 = (1, 0, 0)$
- $V_2 = (1, 1, 0)$
- $V_3 = (1, 0, 1)$

Plotting these points in three-dimensional space reveals a tetrahedron that extends from the origin to points along the axes. The base triangle formed by V_0, V_1, V_2 lies on the xy-plane, while the vertex V_3 extends upward along the z-direction.

Key observations about the shape:

- The base is the triangle with vertices at (0,0,0), (1,0,0), and (1,1,0).
- The apex is at (1,0,1), which is directly above the point (1,0,0).

Boundary equations:

To define the region precisely, we analyze the planes forming the tetrahedron:

1. Plane $z=0$: The base lies on the xy-plane.
2. Plane $x=1$: All points in T satisfy $x \leq 1$.
3. Plane $y=0$: The region extends from $y=0$ to some upper limit.
4. Plane $y= x$: Since the triangle on the base is between $y=0$ and $y=1$, but constrained by the base, the upper boundary for y at a fixed x is $y= x$.
5. Plane $z= x - y$: The top boundary of the tetrahedron can be described by the plane passing through $V_3 = (1, 0, 1)$ and the base.

Let's verify this last plane:

- The points $V_0 = (0,0,0)$, $V_1 = (1,0,0)$, and $V_3 = (1,0,1)$.

Using these points, the plane passing through V_1 and V_3:

- $V_1 = (1,0,0)$
- $V_3 = (1,0,1)$

At $x=1$, z varies from 0 to 1, independent of y. The plane containing V_1 and V_3 with $x=1$ is $z=1$, but to find the general plane, we consider the boundary in terms of x, y, z.

By analyzing the points, a consistent boundary surface for the top of T is:

$$
z = x - y
$$

This is because at $(x,y) = (1,0)$, $z=1 - 0=1$ (matching V_3), and at $(x,y)=(1,1)$, $z=1-1=0$, consistent with the base.

Summary of boundary inequalities:

- $0 \leq x \leq 1$
- $0 \leq y \leq x$
- $0 \leq z \leq x - y$

This description encapsulates the entire tetrahedral volume.

Setting Up the Triple Integral

Given the geometric analysis, the integral over T becomes:

$$
\iiint_T xyzdv = \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x - y} xyzdz\,dy\,dx
$$

This order — integrating with respect to z first, then y, then x — is logical due to the specified bounds.

Step 1: Integrate with respect to z

Since the integrand $xy z$ involves z, and the limits for z are from 0 to $x - y$:

$$
\int_0^{x - y} xy z \, dz = xy \int_0^{x - y} z \, dz = xy \left[\frac{z^2}{2} \right]_0^{x - y} = xy \frac{(x - y)^2}{2}
$$

Step 2: Integrate with respect to y

Now, the integral reduces to:

$$
\int_{x=0}^{1} \left(\int_0^{x} xy \frac{(x - y)^2}{2} dy \right) dx
$$

Note that x acts as a constant within the inner integral. So, the inner integral is:

$$
\frac{x}{2} \int_0^{x} y (x - y)^2 dy
$$

Let’s evaluate $I_y = \int_0^{x} y (x - y)^2 dy$.

Step 3: Calculate I_y

First, expand $(x - y)^2$:

$$
(x - y)^2 = x^2 - 2xy + y^2
$$

Then,

$$
I_y = \int_0^{x} y (x^2 - 2xy + y^2) dy = \int_0^{x} \left(x^2 y - 2x y^2 + y^3 \right) dy
$$

Break into separate integrals:

$$
I_y = x^2 \int_0^{x} y\, dy - 2x \int_0^{x} y^2 dy + \int_0^{x} y^3 dy
$$

Compute each:

- $\int_0^{x} y\, dy = \frac{x^2}{2}$
- $\int_0^{x} y^2 dy = \frac{x^3}{3}$
- $\int_0^{x} y^3 dy = \frac{x^4}{4}$

Plug these back:

$$
I_y = x^2 \left(\frac{x^2}{2} \right) - 2x \left(\frac{x^3}{3} \right) + \frac{x^4}{4} = \frac{x^4}{2} - \frac{2x^4}{3} + \frac{x^4}{4}
$$

Bring to common denominator 12:

$$
I_y = \frac{6x^4}{12} - \frac{8x^4}{12} + \frac{3x^4}{12} = \frac{(6 - 8 + 3) x^4}{12} = \frac{1 x^4}{12} = \frac{x^4}{12}
$$

Now, recalling the inner integral:

\[
\frac{x}{2} \times I_y = \frac{x}{2} \times \frac{x^4}{12} = \frac{x^{5}}{24}
\]

Step 4: Final outer integral over \(x \)

The integral reduces to:

\[
\int_0^{1} \frac{x^5}{24} dx = \frac{1}{24} \int_0^{1} x^5 dx = \frac{1}{24} \left[\frac{x^6}{6} \right]_0^{1} = \frac{1}{24} \times \frac{1}{6} = \frac{1}{144}
\]

Result:

\[
\boxed{
\iiint_T xyzdv = \frac{1}{144}
}
\]

Interpreting the Result and Final Remarks

The computed value \(\frac{1}{144}

[Evaluate Triple Integral Xyzdv Where T Is The Tetrahedron With Vertices 0 0 0 1 0 0 1 1 0 1 0 1]

Evaluate Triple Integral Xyzdv Where T Is The Tetrahedron With Vertices 0 0 0 1 0 0 1 1 0 1 0 1

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