Find The Equation Of The Line Through (0, 2, 1) That Perpendicular To Both U =(4, 3, -5) And The Z-axis.

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Understanding how to find the equation of a line in three-dimensional space that satisfies certain perpendicularity conditions is a fundamental concept in vector calculus and analytical geometry. This article will guide you through the step-by-step process of determining the line passing through a specific point and perpendicular to given vectors, specifically focusing on the point (0, 2, 1), the vector U = (4, 3, -5), and the Z-axis.

Introduction to Lines in Three-Dimensional Space

In three-dimensional geometry, a line can be represented using its parametric equations, which are derived from a point on the line and a direction vector. The general form of a line passing through a point $P_0 = (x_0, y_0, z_0)$ with direction vector $\vec{d} = (a, b, c)$ is:

$$
\begin{cases}
x = x_0 + a t \\
y = y_0 + b t \\
z = z_0 + c t
\end{cases}
$$

where t is a parameter ranging over all real numbers.

Understanding the Problem: Key Components

The problem involves three key elements:

1. The point $P = (0, 2, 1)$ through which the line passes.
2. The vector $\mathbf{U} = (4, 3, -5)$ which the line must be perpendicular to.
3. The Z-axis, represented by the vector $\mathbf{k} = (0, 0, 1)$, which the line must also be perpendicular to.

The main goal is to determine the equation of the line that passes through P and is perpendicular to both $\mathbf{U}$ and the Z-axis.

Understanding Perpendicularity in Space

Perpendicularity (orthogonality) between vectors is key in defining the orientation of the line.

- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if their dot product is zero:

$$
\mathbf{a} \cdot \mathbf{b} = 0
$$

- For the line to be perpendicular to both $\mathbf{U}$ and the Z-axis, its direction vector $\mathbf{d}$ must satisfy:

$$
\mathbf{d} \cdot \mathbf{U} = 0
$$
$$
\mathbf{d} \cdot \mathbf{k} = 0
$$

The first condition ensures perpendicularity to $\mathbf{U}$, and the second ensures perpendicularity to the Z-axis.

Step 1: Identify the Constraints for the Direction Vector

Given the constraints:

1. $\mathbf{d} \cdot \mathbf{U} = 0$:

$$
a \times 4 + b \times 3 + c \times (-5) = 0
$$
which simplifies to:

$$
4a + 3b - 5c = 0
$$

2. $\mathbf{d} \cdot \mathbf{k} = 0$:

$$
a \times 0 + b \times 0 + c \times 1 = 0
$$
which simplifies to:

$$
c = 0
$$

From the second condition, $c = 0$. Substituting this into the first constraint:

$$
4a + 3b - 5 \times 0 = 0 \Rightarrow 4a + 3b = 0
$$

Step 2: Find the Direction Vector $\mathbf{d}$

With $c = 0$, the first constraint reduces to:

$$
4a + 3b = 0
$$

This is a linear relation between a and b. To find a particular solution, choose a parameter:

Let $b = t$, then:

$$
a = - \frac{3}{4} t
$$

Since $c = 0$, the direction vector is:

$$
\mathbf{d} = \left(a, b, c \right) = \left(- \frac{3}{4} t, t, 0 \right)
$$

For simplicity, choose $t = 4$ to clear the denominator:

$$
a = -3, \quad b = 4, \quad c = 0
$$

Therefore, the direction vector:

$$
\mathbf{d} = (-3, 4, 0)
$$

\]

This vector is perpendicular to both \(\mathbf{U} \) and the Z-axis, as required.

Step 3: Write the Parametric Equations of the Line

Using the point \(P = (0, 2, 1) \) and the direction vector \(\mathbf{d} = (-3, 4, 0) \), the parametric equations are:

\[
\begin{cases}
x = 0 - 3t \\
y = 2 + 4t \\
z = 1 + 0 \times t = 1
\end{cases}
\]

This indicates that the line lies in a plane parallel to the XY-plane at \(z = 1 \).

Step 4: Write the Symmetric Equation of the Line

From the parametric equations:

\[
x = -3t \Rightarrow t = -\frac{x}{3}
\]
\[
y = 2 + 4t \Rightarrow t = \frac{y - 2}{4}
\]
\[
z = 1
\]

Since \(t \) must be equal in both expressions:

\[
- \frac{x}{3} = \frac{y - 2}{4}
\]

Rearranged, the symmetric form of the line is:

\[

$$
\frac{x}{-3} = \frac{y - 2}{4} \quad \text{and} \quad z = 1
$$

or equivalently:

$$
\frac{x}{-3} = \frac{y - 2}{4}
$$

This describes the line passing through (0, 2, 1), perpendicular to both the vector $\mathbf{U}$ and the Z-axis.

Summary of the Final Equation

The equation of the line in parametric form:

$$
\boxed{
\begin{cases}
x = -3t \\
y = 2 + 4t \\
z = 1
\end{cases}
}
$$

or in symmetric form:

$$
\frac{x}{-3} = \frac{y - 2}{4}, \quad z = 1
$$

Additional Insights and Applications

Understanding how to find lines perpendicular to given vectors and passing through specific points has practical applications in physics, engineering, computer graphics, and robotics. For example:

- Designing mechanical components: Ensuring parts move along lines perpendicular to certain axes or vectors.
- Computer graphics: Calculating camera angles or object orientations based on vector constraints.
- Navigation and robotics: Planning paths that avoid certain directions or align perpendicularly with obstacles.

Common Mistakes to Avoid

When solving such problems, be cautious of:

- Incorrect vector dot products: Always double-check calculations.
- Sign errors: Pay attention to signs when choosing particular solutions.
- Forgetting the constraints: Remember that the direction vector must satisfy all perpendicularity conditions.
- Misinterpretation of the problem: Clarify whether the line is in space or in a plane, and verify the point and vectors involved.

Conclusion

In summary, to find the equation of a line passing through $(0, 2, 1)$ and perpendicular to both the vector $\mathbf{U} = (4, 3, -5)$ and the Z-axis, we:

- Set up the perpendicularity conditions via dot products.
- Solved for the direction vector satisfying these conditions.
- Used the point and direction vector to write the parametric and symmetric equations.

This thorough approach ensures accuracy and enhances understanding of geometric relations in three-dimensional space. Mastering such methods is vital for advanced studies in vector calculus, physics, and engineering disciplines.

Keywords: Equation of line, 3D geometry, perpendicular vectors, parametric equations, symmetric form, vector calculus, space geometry, analytical geometry

Frequently Asked Questions

How do you find a line passing through the point (0, 2, 1) that is perpendicular to both a given vector U = (4, 3, -5) and the Z-axis?

To find such a line, first determine the direction vector of the line, which must be perpendicular to both U and the Z-axis. Compute the cross product of U and the Z-axis unit vector (0, 0, 1) to get this direction vector, then use the point (0, 2, 1) and this direction vector to write the parametric equations of the line.

What is the cross product of U = (4, 3, -5) and the Z-axis vector (0, 0, 1)?

The cross product U × (0, 0, 1) is calculated as:

|i j k |
|4 3 -5 |
|0 0 1 |

= i(31 - (-5)0) - j(41 - (-5)0) + k(40 - 30)
= i(3) - j(4) + k(0)
= (3, -4, 0).

What is the parametric equation of the line passing through (0, 2, 1) with direction vector (3, -4, 0)?

The parametric equations are:
x = 0 + 3t,
y = 2 - 4t,
z = 1,
where t is a real parameter.

Why is the cross product used to find the direction vector of the line?

The cross product of two vectors yields a vector perpendicular to both, ensuring the line's direction vector is perpendicular to both U and the Z-axis, satisfying the problem's conditions.

What is the final vector form of the line passing through (0, 2, 1)?

The vector form is:
r(t) = (0, 2, 1) + t(3, -4, 0),
where $t \in \mathbb{R}$.

How do you verify that the line is perpendicular to both U and the Z-axis?

Check that the dot product of the line's direction vector (3, -4, 0) with U = (4, 3, -5) is zero, and also with the Z-axis vector (0, 0, 1). For the first, (3)(4) + (-4)(3) + (0)(-5) = 12 - 12 + 0 = 0. For the second, (3)(0) + (-4)(0) + (0)(1) = 0. Both are zero, confirming perpendicularity.

Additional Resources

Find The Equation Of The Line Through (0, 2, 1) That Is Perpendicular To Both U = (4, 3, -5) And The Z-Axis

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Introduction: The Significance of Spatial Lines in Vector Geometry

In the realm of three-dimensional coordinate systems, the study of lines and their orientations relative to vectors and axes is fundamental to understanding the geometric structure of space. Whether in advanced mathematics, engineering, computer graphics, or physics, the ability to determine the equation of a line based on specific constraints is a crucial skill. The problem at hand involves finding the equation of a line passing through a given point and perpendicular to two distinct directions: a vector $\mathbf{U} = (4, 3, -5)$ and the Z-axis. This scenario encapsulates core concepts of vector algebra and spatial reasoning, illustrating how the properties of vectors influence the orientation and positioning of lines in three-dimensional space.

Understanding the Problem: Key Components and Definitions

Before delving into the solution, it is essential to clarify the problem's components:

- Point on the line: $(0, 2, 1)$. This point serves as a known position through which the desired line passes.
- Vector $\mathbf{U} = (4, 3, -5)$: A given vector in space, which the line must be perpendicular to.
- Z-axis: The line passing through points $(0, 0, 0)$ and $(0, 0, 1)$. The Z-axis can be represented by the vector $\mathbf{k} = (0, 0, 1)$.

The core task is to find the equation of a line that:

1. Passes through $(0, 2, 1)$,
2. Is perpendicular to $\mathbf{U}$,
3. Is perpendicular to the Z-axis (represented by $\mathbf{k}$).

Section 1: Mathematical Foundations and Approach

1.1 Vector Equations of a Line

In three-dimensional space, the equation of a line can be expressed in vector form as:

$$
\mathbf{r} = \mathbf{r}_0 + t \mathbf{d}
$$

where:

- $\mathbf{r}_0$ is a point on the line (here, $(0, 2, 1)$),
- $\mathbf{d}$ is the direction vector of the line,
- t is a real parameter.

Our goal is to find the direction vector $\mathbf{d}$ that satisfies the given perpendicularity conditions.

1.2 Perpendicularity Conditions in Vector Algebra

Two vectors are perpendicular if their dot product equals zero:

$$
\mathbf{a} \cdot \mathbf{b} = 0
$$

Applying this principle:

- The line's direction vector $\mathbf{d}$ must satisfy:

$$
\mathbf{d} \cdot \mathbf{U} = 0
$$
and
$$
\mathbf{d} \cdot \mathbf{k} = 0
$$

These conditions ensure the line is perpendicular to both $\mathbf{U}$ and the Z-axis.

Section 2: Deriving the Direction Vector $\mathbf{d}$

2.1 Formulating the Equations from Perpendicularity

Given $\mathbf{d} = (d_x, d_y, d_z)$, the perpendicularity conditions are:

1. $\mathbf{d} \cdot \mathbf{U} = 4d_x + 3d_y - 5d_z = 0$

2. $\mathbf{d} \cdot \mathbf{k} = d_z = 0$

The second condition simplifies the problem significantly:

$$
d_z = 0
$$

This indicates that the direction vector $\mathbf{d}$ lies entirely within the XY-plane, as it has no component along the Z-axis.

2.2 Solving the System for $\mathbf{d}$

With $d_z = 0$, the first equation reduces to:

$$
4d_x + 3d_y = 0
$$

This is a linear relation between d_x and d_y. Let's express d_y in terms of d_x:

$$
3d_y = -4d_x \quad \Rightarrow \quad d_y = -\frac{4}{3} d_x
$$

To have a specific direction vector, assign a free parameter $d_x = t$. Then:

$$
d_y = -\frac{4}{3} t
$$

and

$$
d_z = 0
$$

Hence, the general form of the direction vector becomes:

$$
\mathbf{d} = (t, -\frac{4}{3} t, 0)
$$

Choosing $t = 3$ for simplicity (to clear fractions):

$$
\mathbf{d} = (3, -4, 0)
$$

This vector is perpendicular to $\mathbf{U}$ and the Z-axis, satisfying the problem's conditions.

Section 3: Equation of the Required Line

3.1 Assembling the Equation

Using the point $\mathbf{r}_0 = (0, 2, 1)$ and the direction vector $\mathbf{d} = (3, -4, 0)$, the parametric equations of the line are:

$$
\begin{cases}
x = 0 + 3t \\
y = 2 - 4t \\
z = 1 + 0 \times t = 1
\end{cases}
$$

In vector form, the line can be expressed as:

$$
\boxed{
\mathbf{r}(t) = (0, 2, 1) + t (3, -4, 0)
}
$$

or explicitly:

$$
\mathbf{r}(t) = (3t, 2 - 4t, 1)
$$

where $t \in \mathbb{R}$.

3.2 Geometric Interpretation of the Line

This line is parallel to the XY-plane (since $d_z=0$), ensuring it lies entirely within a plane parallel to the XY-plane, passing through the point $(0, 2, 1)$. Its direction vector indicates movement in the XY-plane with no change in the Z-coordinate, satisfying the perpendicularity constraints.

Section 4: Analytical Validation and Additional Insights

4.1 Confirming Perpendicularity

- Dot product with $\mathbf{U}$:

$$
(3, -4, 0) \cdot (4, 3, -5) = 3 \times 4 + (-4) \times 3 + 0 \times (-5) = 12 - 12 + 0 = 0
$$

- Dot product with the Z-axis vector $\mathbf{k} = (0, 0, 1)$:

$$
(3, -4, 0) \cdot (0, 0, 1) = 0 + 0 + 0 = 0
$$

Both conditions are satisfied, confirming the correctness of the direction vector.

4.2 Uniqueness and Variations

The derived direction vector $(3, -4, 0)$ is one of infinitely many solutions, scaled by any non-zero scalar. For example, choosing $t = 1$, $\mathbf{d} = (1, -\frac{4}{3}, 0)$, yields a similar line. The key is that all such vectors are scalar multiples of the fundamental solution, and they all satisfy the perpendicularity constraints.

Section 5: Broader Applications and Contextual Significance

Understanding how to find lines under multiple perpendicularity constraints has broad applications:

- Engineering and Robotics: Designing robotic arms or components that must move within specific planes or orientations.

- Physics: Analyzing trajectories that are constrained by force vectors or magnetic fields.
- Computer Graphics: Rendering scenes with precise geometric relationships and constraints.
- Mathematics and Education: Enhancing comprehension of vector spaces, dot products, and spatial reasoning.

This problem exemplifies the elegance of vector algebra in solving complex spatial questions, showcasing how algebraic methods translate into geometric insights.

Conclusion: The Synthesis of Geometry and Algebra

The process of determining the equation of a line passing through a point and perpendicular to specified vectors is a testament to the power of vector algebra in three-dimensional geometry. By translating geometric constraints into algebraic equations, we systematically derive the line's direction vector and then formulate its parametric equations. The solution not only satisfies the problem's immediate requirements but also reinforces foundational concepts such as dot product properties, vector parametrization, and spatial reasoning.

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