Find The Flux Of The Vector Field F Across The Surface S In The Indicated ... F=z^4k/16 ; S Is The Upper

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Understanding how to compute the flux of a vector field across a surface is a fundamental concept in vector calculus, with applications spanning physics, engineering, and mathematics. In this article, we will delve into the detailed process of calculating the flux of the vector field $\mathbf{F} = \frac{z^4 \mathbf{k}}{16}$ across a specified surface S, which is described as the "upper" surface—implying a portion of a surface, possibly bounded by certain curves or surfaces in three-dimensional space. Our goal is to provide a comprehensive, step-by-step guide to this problem, ensuring clarity and depth suitable for students and professionals alike.

Understanding the Problem Statement

Before we jump into calculations, it’s essential to interpret the problem correctly.

The Vector Field $\mathbf{F}$

The given vector field is:

$$
\mathbf{F} = \frac{z^4}{16} \mathbf{k}
$$

This indicates that $\mathbf{F}$ has only a component in the z-direction (the $\mathbf{k}$-direction), and its magnitude depends solely on the z-coordinate as $z^4/16$.

Key observations:

- The vector field is vertical, pointing along the z-axis.
- The field varies with height (z), increasing rapidly for larger z.

The Surface S

The surface S is described as the "upper" surface, which typically refers to a surface lying above a certain boundary, such as a disk, a plane, or a curved surface, extending in the positive z-direction.

Note: As the problem statement is incomplete regarding the exact boundary of surface S, we'll consider common scenarios, such as:

- The upper cap of a paraboloid or a sphere.
- A planar surface like $z = f(x, y)$ over a certain domain.
- The upper portion of a surface bounded by a curve in the xy-plane.

In our detailed explanation, we'll assume a typical case: the upper surface of a disk of radius a in the xy-plane, elevated to a certain height or bounded by a curve.

Mathematical Foundations for Surface Flux Calculation

The flux Φ of a vector field $\mathbf{F}$ across a surface S is given by the surface integral:

$$
\Phi = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS
$$

where:

- $\mathbf{n}$ is the unit normal vector to the surface S,
- dS is the differential surface element.

Alternatively, if the surface is parameterized by $\mathbf{r}(u,v)$, the flux becomes:

$$
\Phi = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du\, dv
$$

where:

- D is the domain in the uv-parameter space.

Step-by-Step Procedure to Find the Flux

To compute the flux, follow these systematic steps:

1. Define the Surface S

- Identify the boundary curve or the domain in the xy-plane.
- Choose a parameterization $\mathbf{r}(u,v)$ suitable for the surface.

Example: Suppose S is the upper hemisphere of a sphere of radius a:

$$
x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi
$$
with $\phi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$.

Or, if S is the top surface of a paraboloid $z = 1 - x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$.

2. Choose the Appropriate Parameterization

- For a planar surface, a simple x, y-parameterization works.
- For curved surfaces, use spherical or cylindrical coordinates.

Example: For the upper hemisphere:

$$
\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta, \; a \sin \phi \sin \theta, \; a \cos \phi)
$$

with $\phi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$.

3. Compute the Surface Element $\mathbf{r}_u \times \mathbf{r}_v$ and dS

- Calculate derivatives $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.
- Compute the cross product $\mathbf{r}_u \times \mathbf{r}_v$.
- The magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ gives the differential surface area dS.

Note: The orientation of $\mathbf{r}_u \times \mathbf{r}_v$ determines the direction of the normal vector $\mathbf{n}$. Choose the orientation consistent with the problem's context.

4. Express $\mathbf{F}$ in Terms of the Parameters

- Substitute the parameterization into $\mathbf{F}$:

$$
\mathbf{F}(x(u,v), y(u,v), z(u,v))
$$

- Since $\mathbf{F}$ depends only on z, this simplifies the substitution.

5. Compute the Dot Product $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$

- Perform the dot product at each point in the parameter domain.

6. Set Up and Evaluate the Double Integral

- Integrate over the parameter domain D:

$$
\Phi = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du\, dv
$$

- Use appropriate limits for u, v.

Illustrative Example: Flux Through a Circular Disk

Let's apply these steps to a concrete example: compute the flux of $\mathbf{F} = \frac{z^4}{16}\mathbf{k}$ across the upper surface of a disk $x^2 + y^2 \leq a^2$ at the plane $z=0$, extended upward to some height h.

Assumption: The surface S is the curved surface of the solid region above the disk $x^2 + y^2 \leq a^2$, from $z=0$ up to $z=h$. For simplicity, suppose S is the upper surface of the paraboloid $z = 1 - x^2 - y^2$, over the disk $x^2 + y^2 \leq 1$.

Calculating the Flux for the Paraboloid Surface

Surface Parameterization

Use cylindrical coordinates:

$$
x = r \cos \theta, \quad y = r \sin \theta, \quad z = 1 - r^2
$$

with:

$$
r \in [0, 1], \quad \theta \in [0, 2\pi]
$$

The parameterization:

$$
\mathbf{r}(r, \theta) = (r \cos \theta, r \sin \theta, 1 - r^2)
$$

Compute $\mathbf{r}_r$ and $\mathbf{r}_\theta$

$$
\mathbf{r}_r = (\cos \theta, \sin \theta, -2r)
$$
$$
\mathbf{r}_\theta = (-r \sin \theta, r \cos \theta, 0)
$$

Cross Product $\mathbf{r}_r \times \mathbf{r}_\theta$

$$
\mathbf{r}_r \times \mathbf{r}_\theta =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & -2r \\
-r \sin \theta & r \cos \theta & 0
\end{vmatrix}
$$

Calculating:

$$
\mathbf{r}_r \times \mathbf{r}_\theta =
$$

Frequently Asked Questions

How do I compute the flux of the vector field F = (z^4/16)k across the surface S, which is the upper part of a specified surface?

To compute the flux, set up the surface integral of the vector field's normal component over S: $\Phi = \iint_S F \cdot n \, dS$. Parameterize the surface, find the surface element dS with its normal vector, and evaluate the integral accordingly.

What is the significance of identifying the surface S as the 'upper' part in calculating flux?

Identifying S as the 'upper' part indicates the surface's orientation and limits, which determines the direction of the normal vector used in the flux integral. Usually, the upper surface has a upward-facing normal, simplifying the computation.

How can I apply the Divergence Theorem to find the flux of F across the surface S?

The Divergence Theorem states that the flux across a closed surface equals the triple integral of the divergence over the volume enclosed. If S is part of a closed surface, include the bottom surface to form a closed surface, compute the divergence of F, and evaluate the volume integral.

What are common parameterizations for the surface S if it's, for example, a paraboloid or a plane?

For a paraboloid, you can use cylindrical coordinates: $x = r \cos \theta$, $y = r \sin \theta$, $z = z(r)$. For a plane, a simple x-y parameterization suffices: $r = (x, y)$, with z defined by the plane equation. Choose the parameterization that matches the surface's shape for easier integration.

Are there any simplifications possible for the flux integral when the vector field depends only on z?

Yes, if F depends only on z and points in the z-direction, the dot product with the surface normal simplifies, especially if the surface is aligned with coordinate planes. This can reduce the surface integral to a single-variable integral over the projected domain.

Additional Resources

Find The Flux Of The Vector Field F Across The Surface S In The Indicated ... F=z^4k/16 ; S Is The Upper

Understanding how to compute the flux of a vector field across a surface is a fundamental concept in vector calculus, with applications spanning physics, engineering, and mathematics. In this detailed review, we explore the process of calculating the flux of the vector field F = (z^4/16)k across a specified surface S, particularly focusing on the upper surface of a given region. This task exemplifies key techniques such as parameterization, surface integrals, and the divergence theorem, providing a comprehensive guide for students and practitioners alike.

Introduction to Flux and Its Significance

Flux, in the context of vector calculus, measures the quantity of a vector field passing through a surface. It is a scalar quantity representing the net flow of the field across a boundary, crucial in analyzing physical phenomena like fluid flow, electromagnetic fields, and heat transfer.

Why Calculate Flux?
- To determine how much of a quantity (like fluid, heat, or electromagnetic energy) crosses a surface.
- To apply divergence theorem for converting surface integrals into volume integrals.
- To understand the behavior of fields and their sources or sinks within a region.

The Vector Field F = (z^4/16)k

The given vector field F = (z^4/16)k is a vector field pointing in the z-direction with magnitude depending on the z-coordinate.

Features of F:
- Directionality: Strictly in the z-direction, simplifying the calculation of flux across surfaces aligned or inclined relative to the z-axis.
- Dependence on z: The magnitude varies with z, indicating a non-uniform field that intensifies or weakens depending on the height.
- Applications: Such fields can model scenarios like temperature gradients in the vertical direction or electromagnetic fields aligned vertically.

Understanding the Surface S

The surface S is described as the upper surface of a region. Typically, in calculus problems, the "upper surface" refers to the portion of a boundary above a certain plane or within a particular region.

Common interpretations of "upper surface":
- The top boundary of a solid region, often bounded below by a curve or plane.
- A surface defined by a function $z = f(x, y)$, representing the "upper" boundary above a domain in the xy-plane.

Assumed Description for S:
For the purpose of this review, we consider S as the graph of a function $z = g(x, y)$ over a region D in the xy-plane. For example, it could be $z = \sqrt{a^2 - x^2 - y^2}$ (a hemisphere), or another surface depending on the problem context.

Methodology for Calculating Flux

Calculating flux involves several steps, which we break down systematically:

1. Parameterization of Surface S

To evaluate the surface integral, it's essential to parameterize S. For a graph $z = g(x, y)$, a natural parameterization is:

$$
\vec{r}(x, y) = x \mathbf{i} + y \mathbf{j} + g(x, y) \mathbf{k}
$$

with $(x, y) \in D$, the projection of the surface onto the xy-plane.

2. Computing the Surface Element $d\mathbf{S}$

The oriented surface element vector is given by:

$$
d\mathbf{S} = \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right) dx dy
$$

Calculating this cross product yields a vector normal to the surface, which, when dotted with the vector field F, gives the flux contribution.

3. Dot Product of F with $d\mathbf{S}$

Because F points in the z-direction ($F = (0, 0, z^4/16)$), the dot product simplifies to:
\[
F \cdot d\mathbf{S} = \left(0, 0, \frac{g(x, y)^4}{16}\right) \cdot \left(\text{normal

vector}\right)
\]
This reduces the problem to integrating over the projection domain D.

4. Setting Up and Evaluating the Surface Integral

The flux integral becomes:
\[
\iint_S F \cdot d\mathbf{S} = \iint_D F(x, y, g(x, y)) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right) dx dy
\]

The integral depends on the explicit form of g(x, y) and the region D.

Application of Divergence Theorem

In some cases, directly computing the surface integral is complex. The divergence theorem provides an alternative by converting the surface integral into a volume integral:

\[
\iint_S F \cdot d\mathbf{S} = \iiint_V (\nabla \cdot F) dV
\]

Divergence of F:
\[
\nabla \cdot F = \frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} \left(\frac{z^4}{16}\right) = \frac{4z^3}{16} = \frac{z^3}{4}
\]

This simplifies the calculation if the volume V enclosed by S is known.

Practical Example: Computing Flux for a Specific Surface

Suppose S is the upper hemisphere \(z = \sqrt{a^2 - x^2 - y^2}\), with the projection D being the disk \(x^2 + y^2 \leq a^2\).

Steps:
- Parameterize S using polar coordinates:
\[

$$
x = r \cos \theta,\quad y = r \sin \theta,\quad z = \sqrt{a^2 - r^2}
$$
- Compute the surface element $d\mathbf{S}$.
- Evaluate F at each point:

$$
F = \left(0, 0, \frac{(\sqrt{a^2 - r^2})^4}{16}\right) = \left(0, 0, \frac{(a^2 - r^2)^2}{16}\right)
$$
- Calculate the dot product with $d\mathbf{S}$ and integrate over the disk $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$.

Pros and Cons of the Methods

Surface Integral Approach:
- Pros:
- Directly computes flux across a specific surface.
- Intuitive geometric interpretation.
- Cons:
- Can be algebraically intensive depending on surface complexity.
- Requires parameterization or surface formulas.

Divergence Theorem Approach:
- Pros:
- Transforms surface integrals into volume integrals, often easier to compute.
- Useful when the volume is simple, or divergence is straightforward.
- Cons:
- Requires knowledge of the volume V enclosed by S.
- May involve complex volume integrations.

Summary and Final Remarks

Calculating the flux of the vector field $F = (z^4/16)k$ across an upper surface S involves understanding the geometry of S, parameterizing the surface, and applying surface integration techniques. When feasible, the divergence theorem simplifies the process, especially for symmetric or standard shapes like spheres or cylinders.

This detailed approach emphasizes clarity, step-by-step procedures, and the importance of choosing the most efficient method based on the problem context. Whether tackling theoretical problems or practical applications, mastering these techniques equips students and professionals with powerful tools in vector calculus.

Key Takeaways:

- Always begin by understanding the geometry of the surface and the vector field.
- Use parameterization to simplify the surface integral where possible.
- Consider divergence theorem for complex or symmetric regions.
- Be attentive to the orientation of the surface to ensure correct flux calculation.
- Practice with various surfaces and fields to gain fluency in these techniques.

This comprehensive review aims to serve as a valuable resource for those engaged in vector calculus, providing insights into the methods and considerations involved in flux computations across surfaces.

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